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NON-DENSITY OF SMALL POINTS ON DIVISORS ON ABELIAN VARIETIES AND THE BOGOMOLOV CONJECTURE

KAZUHIKO YAMAKI

1. Introduction

1.1. **Bogomolov conjecture for curves.** Let K be a function field or a number field. Here, a function field means the function field of a normal projective variety of positive dimension over an algebraically closed field k. We fix an algebraic closure $\overline{K}$. We consider the heights on projective varieties over $\overline{K}$ (cf. Section 2.1).

Let C be a smooth projective curve of genus $g \geq 2$ over $\overline{K}$, and let J_C be the Jacobian variety of C. Fix a divisor D on C of degree 1, and let $\jmath_D : C \hookrightarrow J_C$ be the embedding defined by $\jmath_D(x) := x - D$. For any $P \in J_C\left(\overline{K}\right)$ and any $\epsilon \geq 0$, set

$$B_C(P,\epsilon) := \left\{ x \in C\left(\overline{K}\right) \,\middle|\, ||\jmath_D(x) - P||_{NT} \leq \epsilon \right\},$$

where $||\cdot||_{NT}$ is the semi-norm arising from the Néron-Tate height on J_C associated to a symmetric theta divisor. In [1], Bogomolov conjectured the following. Here, in the case where K is a function field, a curve C over $\overline{K}$ is said to be *isotrivial* if it can be defined over the constant field k.

Conjecture 1.1. Assume that C is non-isotrivial when K is a function field. Then, for any $P \in J_C\left(\overline{K}\right)$, there exists $r > 0$ such that $B_C(P, r)$ is a finite set.

When K is a number field, Ullmo proved in 1998 that the conjecture holds (cf. [25, Théorème 1.1]). In the proof, he used the Szpiro-Ullmo-Zhang equidistribution theorem of small points over an archimedean place.

Suppose that K is the function field of a normal projective variety $\mathfrak{B}$ over k. Then the conjecture has been proved for some K but not yet for all K. If $\dim(\mathfrak{B}) = 1$ and $\mathrm{char}(k) = 0$, after partial answers such as [3, 14], Cinkir established the conjecture in [2] by using [34]. For more general K, there are some partial answers such as [12, 27, 28, 32] (under the assumption of $\dim(\mathfrak{B}) = 1$) and [6, 30, 31] (special case of the geometric Bogomolov conjecture for abelian varieties), but we have not yet had a complete answer to the conjecture except for $\dim(\mathfrak{B}) = 1$ and $\mathrm{char}(k) = 0$.

In this paper, we show that the Bogomolov conjecture holds over any function field K.

Theorem 1.2 (Theorem 5.13)**.** *Let K be a function field over any algebraically closed constant field k. Then Conjecture* 1.1 *holds.*

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As an application, we will give an alternative proof of the Manin-Mumford conjecture for curves in positive characteristic (cf. Section 6.2).

Note, however, that Theorem 1.2 is not effective in contrast to Cinkir's theorem (when K is a function field of transcendence degree 1 over k of characteristic 0); Cinkir gave explicitly a positive number r such that $B_C(P,r)$ is finite for any $P \in J_C\left(\overline{K}\right)$ when C has a semistable reduction over K.

1.2. **Non-density of small points of subvarieties of abelian varieties.** We should recall the reason why Cinkir's proof in characteristic 0 does not work in a positive characteristic. His proof is based on Zhang's work [34]. Zhang proved there that if some constants, called the φ-invariants, arising from the reduction graphs of the curve C are positive, and if the height of some cycle of the triple product C^3, called the Gross-Schoen cycle, are non-negative, then the Bogomolov conjecture holds for C. Then Cinkir proved in [2] the positivity of φ-invariants. This suffices for the Bogomolov conjecture in characteristic 0 because in this case it is deduced from the Hodge index theorem that the height of the Gross-Schoen cycle is non-negative. However, in a positive characteristic, the Hodge index theorem, which is a part of the standard conjecture, is not known. Thus the Bogomolov conjecture in a positive characteristic cannot be deduced in the same way.

The proof of Theorem 1.2 is quite different from that of Cinkir's theorem; we do not use reduction graphs of curves or the Gross-Schoen cycles. In our argument, Theorem 1.2 is a direct consequence of a more general result concerning the non-density of small points on curves in abelian varieties, and this result is deduced from the non-density of small points on divisors on abelian varieties, as we will now explain.

Let A be an abelian variety over $\overline{K}$ and let L be an even ample line bundle on A. Here, "even" means the pullback $[-1]^*(L)$ by the (-1)-times automorphism coincides with L. Let X be a closed subvariety of A, and put

$$X(\epsilon; L) := \left\{ x \in X\left(\overline{K}\right) \,\middle|\, \widehat{h}_L(x) \leq \epsilon \right\}.$$

Then we say that X *has dense small points* if $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$. The notion of density of small points is known to be independent of the choice of such L's (cf. [29, Lemma 2.1 and Definition 2.2]).

Let K be a function field and let A be an abelian variety over $\overline{K}$. We recall the notions of $\overline{K}/k$-trace and special subvarieties of A. A *$\overline{K}/k$-trace* of A is a pair $(\widetilde{A}^{\overline{K}/k}, \mathrm{Tr}_A)$ of an abelian variety over k and a homomorphism $\mathrm{Tr}_A : \widetilde{A}^{\overline{K}/k} \otimes_k \overline{K} \to A$ such that Tr_A is the universal one among the homomorphisms to A from abelian varieties which can be defined over the constant field k (see Section 2.1 for more details). It is known that there exists a unique $\overline{K}/k$-trace of A. Using the $\overline{K}/k$-trace, we defined the notion of special subvarieties in [29, Section 2]; a closed subvariety X of A is said to be *special* if there exist a closed subvariety $\widetilde{Y}$ of $\widetilde{A}^{\overline{K}/k}$, an abelian subvariety G of A, and a torsion point $\tau \in A(\overline{K})$ such that $X = \mathrm{Tr}_A(\widetilde{Y} \otimes_k \overline{K}) + G + \tau$.

We will prove the following theorem.

Theorem 1.3 (Theorem 5.12)**.** *Let A be an abelian variety over $\overline{K}$. Let X be a closed subvariety of A with* $\dim(X) = 1$*. Then if X has dense small points, then it is special.*

As is remarked in [30, Section 8], Theorem 1.2 is an immediate consequence of Theorem 1.3.

Theorem 1.3 will be deduced from the following theorem, which is the crucial result in this paper.

Theorem 1.4 (Theorem 5.7)**.** *Let A be an abelian variety over $\overline{K}$. Let X be a closed subvariety of A of codimension* 1*. Then if X has dense small points, then it is special.*

The basic idea to connect Theorem 1.3 to Theorem 1.4 is as follows. Let X be a closed subvariety of A of dimension 1. We set $Y_0 := \{0\}$, and for each positive integer m, let Y_m denote the sum of m copies of $X - X$. Then Y_m is an abelian subvariety of A for large m, and let N be the smallest positive integer among such m. Suppose that X has dense small points. Then we can show that there exists a torsion point $\tau \in A\left(\overline{K}\right)$ such that Y_{N-1} or $Y_{N-1} + (X - \tau)$ is an effective divisor on Y_N, which we denote by D. Note that D also has dense small points. By considering these D and Y_N, we can deduce Theorem 1.3 from Theorem 1.4.

These two theorems concern the following conjecture, called the *geometric Bogomolov conjecture.*

Conjecture 1.5 (Conjecture 2.9 of [29])**.** Let A be an abelian variety over $\overline{K}$. Let X be a closed subvariety of A. Then X has dense small points if and only if X is a special subvariety.

Since a special subvariety has dense small points, the problem is the "only if" part. The geometric Bogomolov conjecture is the geometric version of the Ullmo-Zhang theorem [33], which is called the Bogomolov conjecture for abelian varieties. We refer to Section 6.1 for more details including some background of this conjecture.

As a consequence of Theorems 1.3 and 1.4, we see that the geometric Bogomolov conjecture holds for A with $\dim(A) \leq 3$ (Corollary 6.3), because any non-trivial closed subvariety has dimension 1 or codimension 1 in this case. Furthermore, combining this result with [31, Theorem 1.5] (cf. Theorem 6.2), we see that the conjecture holds for A with $\dim(\mathfrak{m}) \leq \dim(\widetilde{A}^{\overline{K}/k}) + 3$ (Corollary 6.4 and Remark 6.5), where $\mathfrak{m}$ is the maximal nowhere degenerate abelian subvariety of A. See Section 6.1 for more details.

1.3. **Idea.** We describe the idea of the proof of Theorem 1.4. To avoid technical difficulties, we assume $\dim(\mathfrak{B}) = 1$ in this subsection. Let A be an abelian variety of dimension n over $\overline{K}$, and let X be a closed subvariety of A of codimension 1. Remark that X has dense small points if and only if X has canonical height 0 with respect to an even ample line bundle (cf. Proposition 2.1). Then what we should show is that if X is not a special subvariety, then X has a positive canonical height. From Theorem 6.2, which is proved in [30] and [31], together with a few arguments, one sees that it suffices to show the theorem under the assumption that A is nowhere degenerate and has trivial $\overline{K}/k$-trace, where "A is nowhere degenerate" essentially means "A has potentially good reduction everywhere"; see Section 2.1 for the precise terminology. Further, noting that any effective divisor is the pullback of an ample divisor by some homomorphism (cf. [16, page 88, Remarks on effective divisors by Nori]), one finds that it suffices to show the following assertion (cf. Proposition 4.6): Assume that A is a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace; let

X be a subvariety of A of codimension 1 and suppose that X is ample; then X has a positive canonical height.

In this description, we furthermore make the following assumption:

(a) there exists an abelian scheme $f : \mathscr{A} \to \mathfrak{B}$ with zero-section 0_f and with geometric generic fiber A, and X is defined over K;
(b) $0 \notin X$;
(c) $\#k > \aleph_0$, i.e., k has uncountably infinite cardinality.

As we can see later, those assumptions do not give essential restrictions.

Let us give a sketch of the proof. Put $D := X + [-1]^*(X)$, where "+" is the addition of divisors, and put $L := \mathcal{O}_A(D)$. Note that L is even and ample. One sees that $\widehat{h}_L(D) = 2\widehat{h}_L(X)$. Let $\mathscr{D}$ be the closure of D in $\mathscr{A}$. Since $0 \notin D$, $0_f^*(\mathscr{D})$ is a well-defined effective divisor on $\mathfrak{B}$. Set $\mathfrak{L} := \mathcal{O}_{\mathfrak{B}}(0_f^*(\mathscr{D}))$, and set $\mathscr{L} := \mathcal{O}_{\mathscr{A}}(\mathscr{D}) \otimes f^*\mathfrak{L}^{\otimes(-1)}$. Then $0_f^*(\mathscr{L}) = \mathcal{O}_{\mathfrak{B}}$, and one sees that

$$\widehat{h}_L(D) = \deg\left(c_1(\mathscr{L})^{\cdot n} \cdot \mathscr{D}\right) = \deg\left(c_1(\mathscr{L})^{\cdot n} \cdot c_1\left(\mathcal{O}_{\mathscr{A}}(\mathscr{D})\right) \cdot \mathscr{A}\right).$$

Since A has canonical height 0, we have $\deg\left(c_1(\mathscr{L})^{\cdot(n+1)} \cdot \mathscr{A}\right) = 0$ (cf. Remark 2.6). It follows that

$$\begin{aligned}\deg(c_1(\mathscr{L})^{\cdot n} \cdot c_1(\mathcal{O}_{\mathscr{A}}(\mathscr{D})) \cdot \mathscr{A}) &= \deg(c_1(\mathscr{L})^{\cdot(n+1)} \cdot \mathscr{A}) + \deg(c_1(\mathscr{L})^{\cdot n} \cdot f^*c_1(\mathfrak{L}) \cdot \mathscr{A}) \\ &= \deg_L(A) \cdot \deg(\mathfrak{L}).\end{aligned}$$

Since $\deg_L(A) > 0$ by the ampleness of L, it remains to show $\deg(\mathfrak{L}) > 0$. In fact, we see in Lemma 4.5, which is a key lemma, that $\mathfrak{L}$ is non-trivial. Since $\mathfrak{L}$ is effective, we conclude $\deg(\mathfrak{L}) > 0$.

The outline of the proof of the non-triviality of $\mathfrak{L}$ is as follows. We prove the non-triviality by contradiction. Suppose that it is trivial. Then one can show that there exists a finite covering $\mathfrak{B}' \to \mathfrak{B}$ such that the complete linear system $|2\mathscr{D}'|$ on $\mathscr{A}'$ is base-point free (cf. Proposition 4.4), where $f' : \mathscr{A}' \to \mathfrak{B}'$ and $\mathscr{D}'$ are the base-change of f and $\mathscr{D}$ by this $\mathfrak{B}' \to \mathfrak{B}$, respectively. Let $\varphi : \mathscr{A}' \to Z$ be the surjective morphism associated to this complete linear system, where Z is a closed subvariety of the dual space of $|2\mathscr{D}'|$. Remark that for any irreducible curve $\gamma \subset \mathscr{A}'$, $\deg\left(c_1(\mathscr{L}') \cdot \gamma\right) = 0$ if and only if $\varphi(\gamma)$ is a point, where $\mathscr{L}'$ is the pullback of $\mathscr{L}$ to $\mathscr{A}'$. Further, remark that for any $a \in A\left(\overline{K}\right)$, $\widehat{h}_L(a) = 0$ if and only if $\deg\left(c_1(\mathscr{L}') \cdot \Delta_a\right) = 0$, where Δ_a is the closure of a in $\mathscr{A}'$. Since $(\mathscr{L}')^{\otimes 2}$ is relatively ample with respect to f', it follows that φ is finite on any fiber, and since the closure of the set of the height 0 points of $A(\overline{K})$ is dense in $\mathscr{A}'$, we see that φ is not generically finite on $\mathscr{A}'$. Since $\#k > \aleph_0$, it follows that $\mathscr{A}'$ contains uncountably many irreducible curves which are flat over $\mathfrak{B}'$ and are contracted to a point by φ. This means that $A\left(\overline{K}\right)$ has uncountably many points of height 0. On the other hand, since A has trivial $\overline{K}/k$-trace, a point of $A\left(\overline{K}\right)$ has height 0 if and only if it is a torsion, and there are only countably many such points. This is a contradiction.

1.4. **Organization.** This article consists of six sections including this introduction. In Section 2, we recall canonical heights and their description in terms of models when the abelian variety is nowhere degenerate. In Section 3, we prove a version of Bertini's theorem on normal varieties, which will be used in the case where K has transcendence degree more than 1. In Section 4, we prove that an effective ample divisor on a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace has positive canonical height (cf. Proposition 4.6), along the idea explained above. Then, we prove the main results in Section 5. In Section 6, we apply the results of Section 5 to obtain partial answers to the geometric Bogomolov conjecture. We also remark the relationship between the Manin-Mumford conjecture over fields of positive characteristic and the geometric Bogomolov conjecture.

2. Preliminary

2.1. **Notation and convention.** In this paper, a natural number means a strictly positive integer. Let $\mathbb{N}$ denote the set of natural numbers.

Let F be a field, and let X be a scheme over F. For a field extension F'/F, we write $X \otimes_F F' := X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F')$. For a morphism $\phi : X \to Y$ of schemes over F, we write $\phi \otimes_F F' : X \otimes_F F' \to Y \otimes_F F'$ for the base-extension to F'. We call X a *variety* over F if X is a geometrically integral scheme separated and of finite type over F.

Let $f : \mathscr{A} \to S$ be an abelian scheme with zero-section 0_f. For any $n \in \mathbb{Z}$, let $[n] : \mathscr{A} \to \mathscr{A}$ denote the n-times endomorphism. Suppose that S is the spectrum of a field, that is, f is an abelian variety. Let L be a line bundle on this abelian variety. We say that L is *even* if $[-1]^*(L) \cong L$. Remark that in this case, $[n]^*(L) \cong L^{\otimes n^2}$ holds by the theorem of the cube (cf. [16, Section 6, Corollary 3]).

Let k be an algebraically closed field, let $\mathfrak{B}$ be a normal projective variety of dimension $b \geq 1$ over k, and let $\mathcal{H}$ be an ample line bundle on $\mathfrak{B}$. Let K be the function field of $\mathfrak{B}$, and let $\overline{K}$ be an algebraic closure of K. All of them are fixed throughout this article (except in Section 6.2). Any finite extension of K will be taken in $\overline{K}$.

An abelian variety B over $\overline{K}$ is called a *constant abelian variety* if there exists an abelian variety $\widetilde{B}$ over k such that $B = \widetilde{B} \otimes_k \overline{K}$ as abelian varieties.

Let A be an abelian variety over $\overline{K}$. A pair $(\widetilde{A}^{\overline{K}/k}, \mathrm{Tr}_A)$ consisting of an abelian variety $\widetilde{A}^{\overline{K}/k}$ over k and a homomorphism $\mathrm{Tr}_A : \widetilde{A}^{\overline{K}/k} \otimes_k \overline{K} \to A$ of abelian varieties over $\overline{K}$ is called a *$\overline{K}/k$-trace* of A if for any abelian variety $\widetilde{B}$ over k and a homomorphism $\phi : \widetilde{B} \otimes_k \overline{K} \to A$, there exists a unique homomorphism $\mathrm{Tr}(\phi) : \widetilde{B} \to \widetilde{A}^{\overline{K}/k}$ such that ϕ factors as $\phi = \mathrm{Tr}_A \circ \big(\mathrm{Tr}(\phi) \otimes_k \overline{K}\big)$. It is unique by the universality, and it is also known to exist. We call Tr_A the *trace homomorphism* of A. See [9] for more details.

Let M_K be the set of points of $\mathfrak{B}$ of codimension 1. For any $v \in M_K$, the local ring $\mathcal{O}_{\mathfrak{B},v}$ is a discrete valuation ring with fractional field K. The order function on $\mathcal{O}_{\mathfrak{B},v}$ extends to a unique order function $\mathrm{ord}_v : K^\times \to \mathbb{Z}$. Recall that we have a fixed ample line bundle $\mathcal{H}$ on $\mathfrak{B}$. Then we have a non-archimedean value $|\cdot|_{v,\mathcal{H}}$ on K normalized in such a way that

$$|x|_{v,\mathcal{H}} := e^{-\deg_{\mathcal{H}}(\overline{v})\,\mathrm{ord}_v(x)}$$

for any $x \in K^\times$, where $\deg_{\mathcal{H}}(\overline{v})$ denotes the degree with respect to $\mathcal{H}$ of the closure $\overline{v}$ of v in $\mathfrak{B}$. It is well known that the set $\{|\cdot|_{v,\mathcal{H}}\}_{v \in M_K}$ of values satisfies the product formula, and hence the notion of (absolute logarithmic) heights with respect to this set of absolute values is defined (cf. [10, Chapter 3 Section 3]).

We recall the notion of places of $\overline{K}$ and notation introduced in [30, Section 6.1]. For a finite extension K' of K in $\overline{K}$, let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K', and let $M_{K'}$ be the set of points of $\mathfrak{B}'$ of codimension 1. An element in $M_{K'}$ is called a *place of K'*. For a finite extension K''/K', we have a natural surjective map $M_{K''} \to M_{K'}$, and thus we obtain an inverse system $(M_{K'})_{K'}$, where K' runs through the finite extensions of K in $\overline{K}$. Set $M_{\overline{K}} := \varprojlim_{K'} M_{K'}$. We call an element of $M_{\overline{K}}$ a *place of* $\overline{K}$. Each $v \in M_{\overline{K}}$ gives a unique absolute value on $\overline{K}$ which extends $|\cdot|_{v_K, \mathcal{H}}$, where v_K is the image of v by the canonical map $M_{\overline{K}} \to M_K$. We denote by $\overline{K}_v$ the completion of $\overline{K}$ with respect to that absolute value, and we let $\overline{K}_v^\circ$ denote the ring of integers of $\overline{K}_v$.

Let A be an abelian variety over $\overline{K}$. Let $v \in M_{\overline{K}}$ be a place. We say that A is *non-degenerate at* v if there exists an abelian scheme over $\overline{K}_v^\circ$ whose generic fiber equals A. We say A is *nowhere degenerate* if A is non-degenerate at any $v \in M_{\overline{K}}$. Those definitions are compatible with the terminology used in [30, 31].

We give a remark on our terminology of nowhere-degeneracy. Suppose that K' is a finite extension of K and that A' is an abelian variety over K' with $A = A' \otimes_{K'} \overline{K}$. Let $v \in M_{\overline{K}}$, and let $v_{K'}$ be the image of v by the natural map $M_{\overline{K}} \to M_{K'}$. Then A is non-degenerate at v if and only if A' has a potentially good reduction at $v_{K'}$, and thus A is nowhere degenerate if and only if A' has a potentially good reduction everywhere.

Let A be an abelian variety over $\overline{K}$, and let L be an even ample line bundle. Then there exists a unique height function $\widehat{h}_L$ on A associated to L such that $\widehat{h}_L$ is a quadratic form on the additive group $A\left(\overline{K}\right)$. This is called the *canonical height* associated to L.

2.2. **Canonical heights.** On an abelian variety, we have a notion of canonical heights not only for points but also for positive dimensional cycles. Let $L_0, \ldots, L_d$ be line bundles on an abelian variety A, and let Z be a cycle of dimension d on A. Then we consider a real number called the *canonical height* of Z with respect to $L_0, \ldots, L_d$, which is denoted by $\widehat{h}_{L_0,\ldots,L_d}(Z)$. It is known that the assignment $(L_0, \ldots, L_d, Z) \mapsto \widehat{h}_{L_0,\ldots,L_d}(Z)$ is multilinear. When $L_0 = \cdots = L_d = L$ and there is no danger of confusion, we simply write $\widehat{h}_L(Z)$ instead of $\widehat{h}_{L_0,\ldots,L_d}(Z)$. We consider the canonical height of a closed subvariety X of A of pure dimension d by regarding X as a cycle in a natural way. We refer to [5–7] for more details.

The canonical heights of subvarieties are important in the study of the density of small points because of the following proposition.

Proposition 2.1 (Corollary 4.4 in [6])**.** Let A be an abelian variety over $\overline{K}$, let L be an even ample line bundle on A, and let X be a closed subvariety of A. Then X has dense small points if and only if $\widehat{h}_L(X) = 0$.

In the case where A is a nowhere degenerate abelian variety, the canonical height of a cycle can be described in terms of intersection products on models. We refer to [4] for intersection theory. First, we recall the notion of models. Let X be a projective scheme over $\overline{K}$, and let L be a line bundle on X. Let K' be a finite extension of K, and let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'. Let $\mathfrak{U}$ be an open subset of $\mathfrak{B}'$. A proper morphism $f : \mathscr{X} \to \mathfrak{U}$ with geometric generic fiber X is called a *model* of X over $\mathfrak{U}$. Furthermore, let $\mathscr{L}$ be a line bundle on $\mathscr{X}$ whose restriction to the geometric generic fiber X equals L. Then the pair $(f, \mathscr{L})$ is called a *model* of (X, L) over $\mathfrak{U}$.

For a nowhere degenerate abelian variety A over $\overline{K}$ and an even line bundle L on A, the following proposition gives us a "normalized" model of (A, L). This is a starting point to describe the canonical height by intersection on models.

Proposition 2.2 (Proposition 2.5 of [31])**.** Let A be a nowhere degenerate abelian variety over $\overline{K}$, and let L be a line bundle on A. Then there exists a finite extension K' of K that satisfies the following condition: Let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K'; then there exist an open subset $\mathfrak{U}$ of $\mathfrak{B}'$ with $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$, an abelian scheme $f : \mathscr{A} \to \mathfrak{U}$ with zero-section 0_f, and a line bundle $\mathscr{L}$ on $\mathscr{A}$ such that $(f, \mathscr{L})$ is a model of (A, L) over $\mathfrak{U}$ with $0_f^*(\mathscr{L}) \cong \mathcal{O}_{\mathfrak{U}}$.

Remark that $\mathfrak{U}$ as well as $\mathfrak{B}'$ is normal and that we have a natural finite surjective morphism $\mathfrak{B}' \to \mathfrak{B}$.

Remark 2.3. We constructed in [31, Proposition 2.5] a model $\left(\bar{f} : \bar{\mathscr{A}} \to \mathfrak{B}', \bar{\mathscr{L}}\right)$ over $\mathfrak{B}'$ whose restriction over $\mathfrak{U}$ coincides with $(f, \mathscr{L})$ as in Proposition 2.2. Such an $\left(\bar{f}, \bar{\mathscr{L}}\right)$ is constructed from $(f, \mathscr{L})$ by using Nagata's embedding theorem. We refer to [26, Theorem 5.7] for a scheme-theoretic proof of this embedding theorem.

Remark 2.4. Proposition 2.2 says in particular that if A is nowhere degenerate, then there exist a finite extension K' of K, an open subset $\mathfrak{U}$ of $\mathfrak{B}'$ with $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$ where $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in K', and an abelian scheme $f : \mathscr{A} \to \mathfrak{U}$ with geometric generic fiber A. Remark that the converse of this also holds; if K' is a finite extension of K, $\mathfrak{U}$ is an open subset of $\mathfrak{B}'$ with $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$ where $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in K', and if $f : \mathscr{A} \to \mathfrak{U}$ is an abelian scheme, then the geometric generic fiber A of f is a nowhere degenerate abelian variety. Indeed, in this setting, since $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in K', the set of places $M_{K'}$ equals the set of codimension 1 points in $\mathfrak{B}'$. Since $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$, that equals the set of codimension 1 points in $\mathfrak{U}$. It follows that the generic fiber of f is an abelian variety that has good reduction at any $v_{K'} \in M_{K'}$, and thus A is nowhere degenerate.

Using a model as in Proposition 2.2, we describe the canonical height of a cycle in terms of intersection by [7, Theorem 3.5 (d)]. We show the following lemma, which will follow easily from that theorem, where we recall $b := \dim(\mathfrak{B})$ and $\mathcal{H}$ is a fixed ample line bundle on $\mathfrak{B}$.

Lemma 2.5. *Let A be a nowhere degenerate abelian variety over $\overline{K}$, and let L be an even line bundle on A. Let X be a closed subscheme of A of pure dimension d. Let K', $\mathfrak{B}'$, $\mathfrak{U}$, $f; \mathscr{A} \to \mathfrak{U}$, 0_f, and $\mathscr{L}$ be as in Proposition 2.2. Let $\mathcal{H}'$ be the pullback of $\mathcal{H}$ by the finite morphism $\mathfrak{B}' \to \mathfrak{B}$. Let $\mathscr{X}$ be the closure of X in $\mathscr{A}$. Let $\mathfrak{D}$ be a $(b-1)$-cycle on $\mathfrak{B}'$ such that $[\mathfrak{D} \cap \mathfrak{U}] = f_*\left(c_1(\mathscr{L})^{\cdot(d+1)} \cdot [\mathscr{X}]\right)$ as cycle classes on $\mathfrak{U}$. Then*

$$\widehat{h}_L(X) = \frac{\deg_{\mathcal{H}'}[\mathfrak{D}]}{[K' : K]}.$$

Proof. As in Remark 2.3, we take a proper morphism $\bar{f} : \bar{\mathscr{A}} \to \mathfrak{B}'$ and a line bundle $\bar{\mathscr{L}}$ such that f is the pullback of $\bar{f}$ by the open immersion $\mathfrak{U} \hookrightarrow \mathfrak{B}'$ and $\bar{\mathscr{L}}|_{\mathscr{A}} = \mathscr{L}$. Let $\bar{\mathscr{X}}$ be the closure of $\mathscr{X}$ in $\bar{\mathscr{A}}$.

Note that

$$\widehat{h}_L(X) = \frac{\deg_{\mathcal{H}'} \bar{f}_*\left(c_1\left(\bar{\mathscr{L}}\right)^{\cdot(d+1)} \cdot \left[\bar{\mathscr{X}}\right]\right)}{[K' : K]}. \tag{2.5.1}$$

Indeed, since $\mathfrak{U}$ is normal, $f : \mathscr{A} \to \mathfrak{U}$ is projective by [19, Théorème XI 1.4]. By [22, Corollaire 5.7.14] or by [26, Corollary 2.6], there exists a proper morphism $\mu : \bar{\mathscr{A}}^\dagger \to \bar{\mathscr{A}}$ isomorphic over $\mathscr{A}$ such that $\bar{f}^\dagger := \bar{f} \circ \mu$ is projective and such that $\mu^{-1}(\mathscr{A})$ is scheme-theoretically dense in $\bar{\mathscr{A}}^{\dagger}$. Noting that $\bar{f}^{\dagger} : \bar{\mathscr{A}}^{\dagger} \to \mathfrak{B}'$ is also a model of A, we let $\bar{\mathscr{X}}^{\dagger}$ be the closure of X in $\bar{\mathscr{A}}^{\dagger}$. Since $\bar{f}^{\dagger}$ is projective, [7, Theorem 3.5 (d)] gives us

$$\widehat{h}_L(X) = \frac{\deg_{\mathcal{H}'} \bar{f}^{\dagger}_{*}\left(c_1\left(\mu^{*}\left(\bar{\mathscr{L}}\right)\right)^{\cdot(d+1)} \cdot \left[\bar{\mathscr{X}}^{\dagger}\right]\right)}{[K' : K]}.$$

By the projection formula, we thus obtain (2.5.1).

Since $\bar{f}_{*}(c_1(\bar{\mathscr{L}})^{\cdot(d+1)} \cdot [\bar{\mathscr{X}}])$ is a cycle class on $\mathfrak{B}'$ of codimension 1 whose restriction to $\mathfrak{U}$ equals $f_{*}(c_1(\mathscr{L})^{\cdot(d+1)} \cdot [\mathscr{X}])$ and since $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$, we have

$$[\mathfrak{D}] = \bar{f}_{*}\left(c_1\left(\bar{\mathscr{L}}\right)^{\cdot(d+1)} \cdot \left[\bar{\mathscr{X}}\right]\right)$$

as cycle classes. Thus the lemma follows from (2.5.1). □

Remark 2.6. Let $\bar{f} : \bar{\mathscr{A}} \to \mathfrak{B}'$ be as in Remark 2.3. Then

$$\deg_{\mathcal{H}'} \bar{f}_{*}\left(c_1\left(\bar{\mathscr{L}}\right)^{\cdot(\dim(A)+1)} \cdot \left[\bar{\mathscr{A}}\right]\right) = 0.$$

Indeed, since A has dense small points, it has canonical height 0 (cf. Proposition 2.1), and hence the equality follows from Lemma 2.5 (or (2.5.1)).

3. Bertini-type theorem

The purpose of this section is to show Proposition 3.5, which concerns curves that are intersections of general hyperplanes on a projective variety. This proposition will be applied later to a finite covering of $\mathfrak{B}$ in the case of $b := \dim(\mathfrak{B}) \geq 2$.

Let $\mathcal{O}(1)$ denote the tautological line bundle of $\mathbb{P}^N_k$, and let $|\mathcal{O}(1)|$ denote the complete linear system of hyperplanes on $\mathbb{P}^N_k$. Let X be a subvariety of $\mathbb{P}^N_k$. Let r be a natural number. For any $\underline{H} = (H_1, \ldots, H_r) \in |\mathcal{O}(1)|^r$, we write $X_{\underline{H}} := X \cap H_1 \cap \cdots \cap H_r$.

Lemma 3.1. *Let X be an irreducible projective scheme of dimension d over k with a closed embedding $X \subset \mathbb{P}^N_k$. Suppose that there exists a regular open subscheme U with $\operatorname{codim}(X \setminus U, X) \geq 2$. Let r be a positive integer with $r \leq d-1$. Then there exists a dense open subset $V(k) \subset |\mathcal{O}(1)|^r$ such that any $\underline{H} = (H_1, \ldots, H_r) \in V(k)$ satisfies the following:*

(a) *$X_{\underline{H}}$ is irreducible and has dimension $d - r$;*
(b) *$U_{\underline{H}}$ is regular;*
(c) $\operatorname{codim}(X_{\underline{H}} \setminus U, X_{\underline{H}}) \geq 2$.

Proof. Note that the assignment $(H_1, \ldots, H_r) \mapsto H_1 \cap \cdots \cap H_r$ defines a dominant rational map from $|\mathcal{O}(1)|^r$ to the grassmannian variety of codimension r linear subspaces of $\mathbb{P}^N_k$. Then the lemma follows from [8, Corollary 6.11] immediately. Indeed, by [8, Corollary 6.11], there exists a dense open subset $V_1(k)$ of $|\mathcal{O}(1)|^r$ such that for any $\underline{H} \in V_1(k)$, $X_{\underline{H}}$ is an irreducible subscheme of dimension $d-r$ and $U_{\underline{H}}$ is regular. Further by [8, Corollary 6.11 (1) (b)], there exists a dense open subset $V_2(k)$ of $|\mathcal{O}(1)|^r$ such that for any $\underline{H} \in V_1(k)$, $\operatorname{codim}\left((X \setminus U)_{\underline{H}}, X\right) \geq 2 + r$. Putting $V(k) := V_1(k) \cap V_2(k)$, we see that any $\underline{H} \in V(k)$ satisfies conditions (a), (b), and (c). □

We sometimes have to consider the linear system consisting of hyperplanes which pass through a specified point. For any $x \in X(k)$, we set

$$|\mathcal{O}(1)|_x^r := \{(H_1, \ldots, H_r) \in |\mathcal{O}(1)|_x^r \mid x \in H_1 \cap \cdots \cap H_r\}.$$

Further, for an open subset $V(k)$ of $|\mathcal{O}(1)|^r$, we set $V_x(k) := V(k) \cap |\mathcal{O}(1)|_x^r$.

Lemma 3.2. *Let d be an integer with $d \geq 2$. Let r be a positive integer with $r \leq d-1$, and let $V(k)$ be a dense open subset of $|\mathcal{O}(1)|^r$. Let $x \in X(k)$ be a point with $V_x(k) \neq \emptyset$. Suppose that W is a non-empty closed subset of $\mathbb{P}_k^N$ with* $\dim(W) \leq d-1$. *Then there exists an $\underline{H} = (H_1, \ldots, H_r) \in V_x(k)$ such that* $\dim(W_{\underline{H}}) \leq d-r-1$. *(Remark that the dimension of a closed subset is the maximum of the dimensions of its irreducible components. Remark also that* $\dim(\emptyset) := -1$ *by convention.)*

Proof. We prove the lemma by induction on r. First let $r = 1$. If $\dim(W) \leq d-2$, then $\dim(W \cap H) \leq d-2$ for any hyperplane H. Therefore we may assume $\dim(W) = d-1$. Let $q_1, \ldots, q_m$ be the generic points of the irreducible components of W of dimension $d-1$. Since $d-1 \geq 1$, no q_i equals x. Therefore

$$U(k) := \{H \in |\mathcal{O}(1)|_x \mid q_1 \notin H, \ldots, q_m \notin H\}$$

is a dense open subset of $|\mathcal{O}(1)|_x$, and hence there exists an $H \in V_x(k) \cap U(k)$. We then have $\dim(W \cap H) \leq d-2$, and thus we have the lemma for $r = 1$.

Suppose that we have the assertion up to r ($1 \leq r \leq d-2$), and we are going to show it for $r+1$. Let $p : |\mathcal{O}(1)|_x^{r+1} \to |\mathcal{O}(1)|_x^r$ be the map given by $p(H_1, \ldots, H_r, H_{r+1}) = (H_1, \ldots, H_r)$. Then the image $p(V_x(k))$ is a dense open subset of $|\mathcal{O}(1)|_x^r$. By the induction hypothesis, there exists an $\underline{H} = (H_1, \ldots, H_r) \in p(V_x(k))$ such that $\dim(W_{\underline{H}}) \leq d-r-1$. Since $p^{-1}(\underline{H}) \cap V_x(k) \neq \emptyset$, this set is a dense open subset of $\{\underline{H}\} \times |\mathcal{O}(1)|_x$. Applying the lemma for $r = 1$ to $W_{\underline{H}}$, we obtain an $H_{r+1} \in p^{-1}(\underline{H}) \cap V_x(k)$ such that $\dim(W_{\underline{H}} \cap H_{r+1}) \leq d-r-2$. This shows the assertion for $r+1$. Thus we obtain the lemma. $\square$

Using Lemma 3.2, we obtain the following.

Lemma 3.3. *Let X be an irreducible projective scheme of dimension d over k with a closed embedding $X \subset \mathbb{P}_k^N$. Let r be a positive integer with $r \leq d-1$, and let $V(k)$ be a dense open subset of $|\mathcal{O}(1)|^r$. Further, let $x \in X(k)$ be a point such that $V_x(k) \neq \emptyset$. Then for any proper closed subset W of X, there exists an $\underline{H} = (H_1, \ldots, H_r) \in V_x(k)$ such that* $\operatorname{Supp}(X_{\underline{H}}) \not\subseteq W$.

Proof. Remark that $d \geq r+1 \geq 2$. Remark also that $\dim(X_{\underline{H}}) \geq d-r$ for any $\underline{H} \in |\mathcal{O}(1)|^r$. Let W be any proper closed subset of X. Then $\dim(W) \leq d-1$. By Lemma 3.2, there exists an $\underline{H} = (H_1, \ldots, H_r) \in V_x(k)$ such that $\dim(W_{\underline{H}}) \leq d-r-1$. Since $\dim(X_{\underline{H}}) \geq d-r$, we have $X_{\underline{H}} \not\subseteq W_{\underline{H}}$, and thus $\operatorname{Supp}(X_{\underline{H}}) \not\subseteq W$. $\square$

We show one more lemma.

Lemma 3.4. *Let X be an irreducible closed subscheme of dimension d of $\mathbb{P}_k^N$. Suppose that there exists a regular open subscheme U of X with* $\operatorname{codim}(X \setminus U, X) \geq 2$. *Let r be a positive integer with $r \leq d-1$, and let $V(k)$ be a dense open subset of $|\mathcal{O}(1)|^r$. Then there exists a dense open subset $U' \subset X$ such that for any $x \in U'(k)$, there exists an $(H_1, \ldots, H_r) \in V(k)$ with $x \in H_1 \cap \cdots \cap H_r$.*

Proof. Set

$$\mathcal{Z}(k) := \{(x, H_1, \dots, H_r) \in X(k) \times V(k) \mid x \in H_1 \cap \cdots \cap H_r\},$$

which is a non-empty closed subset of $X(k) \times V(k)$. Let $g : \mathcal{Z}(k) \to X(k)$ be the restriction of the canonical projection $X(k) \times V(k) \to X(k)$. We take an $x \in g(\mathcal{Z}(k))$. Note that

$$V_x(k) := \{\underline{H} = (H_1, \dots, H_r) \in V(k) \mid x \in H_1 \cap \cdots \cap H_r\} \neq \emptyset.$$

Let $W(k)$ be the closure of $g(\mathcal{Z}(k))$ in $X(k)$. Then we claim $W(k) = X(k)$ by contradiction. Indeed, if this is not the case, then $W(k) \subsetneq X(k)$, and hence Lemma 3.3 gives us an $\underline{H} = (H_1, \dots, H_r) \in V_x(k)$ such that $X_{\underline{H}}(k) \not\subseteq W(k)$. Now we can take a point x' of $X_{\underline{H}}(k) \setminus W(k)$. Then $x' \in g(\mathcal{Z}(k))$ by the definition of $\mathcal{Z}(k)$. It follows that $x' \in g(\mathcal{Z}(k)) \setminus W(k) \subset W(k) \setminus W(k) = \emptyset$, which is a contradiction.

Therefore $g : \mathcal{Z}(k) \to X(k)$ is a dominant morphism. By Chevalley's theorem, there exists a non-empty open subset U' of X such that $U'(k) \subset g(\mathcal{Z}(k))$. Then U' satisfies the required condition, and this completes the proof. $\square$

As a consequence of the arguments so far, we have the following proposition.

Proposition 3.5. *Let X be an irreducible projective scheme of dimension d over k, and let $\mathscr{L}$ be a very ample line bundle on X. Let $|\mathscr{L}|$ denote the complete linear system associated to $\mathscr{L}$. Suppose that there exists a regular open subscheme U of X such that* $\operatorname{codim}(X \setminus U, X) \geq 2$. *Then the following hold.*

(1) *There exists a dense open subset $V(k) \subset |\mathscr{L}|^{d-1}$ such that for any $(D_1, \dots, D_{d-1}) \in V(k)$, $C := D_1 \cap \cdots \cap D_{d-1}$ is an irreducible projective non-singular curve contained in U.*

(2) *Let $V(k)$ be as in (1). For any $x \in X(k)$, set*

$$|\mathscr{L}|_x^{d-1} := \left\{ (D_1, \dots, D_{d-1}) \in |\mathscr{L}|^{d-1} \;\middle|\; x \in D_1 \cap \cdots \cap D_{d-1} \right\}$$

and $V_x(k) := V \cap |\mathscr{L}|_x^{d-1}$. Then there exists a dense open subset $U' \subset U$ such that for any $x \in U'(k)$, $V_x(k)$ is a dense open subset of $|\mathscr{L}|_x^{d-1}$.

(3) *Furthermore, let U' be as in (2). Take any $x \in U'(k)$. Then for any proper closed subset W of X, there exists a $(D_1, \dots, D_{d-1}) \in V_x(k)$ such that* $\operatorname{Supp}(D_1 \cap \cdots \cap D_{d-1}) \not\subseteq W$.

Proof. Let $X \hookrightarrow \mathbb{P}_k^N$ be the closed embedding associated to the complete linear system $|\mathscr{L}|$. Then we have $|\mathscr{L}| = |\mathcal{O}(1)|$, where $\mathcal{O}(1)$ is the tautological line bundle of $\mathbb{P}_k^N$. Now, we see that (1) follows from Lemma 3.1 for $r = d - 1$. Since $V_x(k)$ is open in $|\mathscr{L}|_x^{d-1}$, (2) follows from Lemma 3.4. Finally, (3) follows from Lemma 3.3. $\square$

4. Positivity of the canonical height

In this section, we prove Proposition 4.6, which claims the positivity of the canonical height of an effective ample divisor on a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace.

Let $\mathscr{A} \to S$ be an abelian scheme. We use the following notation. For a section σ of it, we have the translate morphism by σ, which is denoted by $T_\sigma : \mathscr{A} \to \mathscr{A}$. For an n-times endomorphism $[n] : \mathscr{A} \to \mathscr{A}$, we set $\mathscr{A}[n] := \operatorname{Ker}[n]$. That is a finite scheme over S.

4.1. **Translates of effective divisors on abelian varieties.** In this subsection, let A be an abelian variety over k, and let D be an effective divisor on A. We show two lemmas concerning translates of D.

Lemma 4.1. *For any $a \in A(k)$, there exists a dense open subset $V_a \subset A$ such that $a \notin \mathrm{Supp}\left(T_\tau^*(D) + T_{-\tau}^*(D)\right)$ for any $\tau \in V_a(k)$, where "+" means the sum of the divisors.*

Proof. We see that $a \in \mathrm{Supp}\,(T_\tau^*(D))$ is equivalent to $a \in D - \tau$, and this is equivalent to $\tau \in \mathrm{Supp}\,(T_a^*(D))$. This is a closed condition for $\tau \in A(k)$. Similarly, we see that $a \in \mathrm{Supp}\left(T_{-\tau}^*(D)\right)$ is a closed condition for τ. Hence $a \notin \mathrm{Supp}\,(T_\tau^*(D)) \cup \mathrm{Supp}\left(T_{-\tau}^*(D)\right)$ is an open condition for τ. Since $\mathrm{Supp}\left(T_a^*(D) + T_{-\tau}^*(D)\right) \subset \mathrm{Supp}\,(T_\tau^*(D)) \cup \mathrm{Supp}\left(T_{-\tau}^*(D)\right) \subsetneq A$, that shows the existence of a desired dense open subset V_a. $\square$

Lemma 4.2. *Let l be a prime number with $l \neq \mathrm{char}(k)$. Then there exists an $m \in \mathbb{N}$ such that*

$$\bigcap_{\tau \in A[l^m]} \mathrm{Supp}\left(T_\tau^*(D) + T_{-\tau}^*(D)\right) = \emptyset.$$

Proof. For each $n \in \mathbb{N}$, we put

$$S_n := \bigcap_{\tau \in A[l^n]} \mathrm{Supp}\left(T_\tau^*(D) + T_{-\tau}^*(D)\right).$$

Since $(S_n)_{\in \mathbb{N}}$ is a descending sequence of closed subsets of A, there exists an $m \in \mathbb{N}$ such that $S_n = S_m$ for any $n \geq m$. What we should show is that $S_m = \emptyset$. To show this by contradiction, suppose that $S_m \neq \emptyset$. Since S_m is closed, we then take a closed point $s \in S_m(k)$. Let V_s be a dense open subset of A as in Lemma 4.1. Since $l \neq \mathrm{char}(k)$, $\bigcup_{n \in \mathbb{N}} A[l^n]$ is dense in A, and hence we take a point $\tau_0 \in V_s(k) \cap \left(\bigcup_{n \in \mathbb{N}} A[l^n]\right)$. Then $s \notin \mathrm{Supp}\left(T_{\tau_0}^*(D) + T_{-\tau_0}^*(D)\right)$. On the other hand, there exists an $m_0 \in \mathbb{N}$ with $m_0 \geq m$ such that $\tau_0 \in A[l^{m_0}]$. Since $s \notin \mathrm{Supp}\left(T_{\tau_0}^*(D) + T_{-\tau_0}^*(D)\right)$, it follows that

$$s \notin \bigcap_{\tau \in A[l^{m_0}]} \mathrm{Supp}\left(T_\tau^*(D) + T_{-\tau}^*(D)\right) = S_{m_0}.$$

However, since we have taken m and s so that $S_{m_0} = S_m$ and $s \in S_m$, that is a contradiction. Thus we conclude that $S_m = \emptyset$, which completes the proof of the lemma. $\square$

4.2. **Base-point freeness on abelian schemes.** The purpose of this subsection is to establish Proposition 4.4, the base-point freeness on an abelian scheme of an effective even line bundle which is normalized to be trivial along the zero-section.

First, we show the following lemma, which generalizes Lemma 4.2 on abelian schemes.

Lemma 4.3. *Let $\mathfrak{U}$ be an irreducible noetherian scheme, let $f : \mathscr{A} \to \mathfrak{U}$ be an abelian scheme, and let $\mathscr{D}$ be an effective Cartier divisor on $\mathscr{A}$ flat over $\mathfrak{U}$. Let l be a prime number. Suppose that l does not equal the characteristic of the residue field at any $u \in \mathfrak{U}$. Then there exist an $m \in \mathbb{N}$ and a finite étale morphism $\mathfrak{U}' \to \mathfrak{U}$*

with $\mathfrak{U}'$ irreducible that satisfy the following conditions: Let $f' : \mathscr{A}' \to \mathfrak{U}'$ and $\mathscr{D}'$ be the base-change of f and $\mathscr{D}$, respectively, by the morphism $\mathfrak{U}' \to \mathfrak{U}$; then we have

$$\bigcap_{\tau \in \mathscr{A}'[l^m](\mathfrak{U}')} \operatorname{Supp}\left(T_\tau^*(\mathscr{D}') + T_{-\tau}^*(\mathscr{D}')\right) = \emptyset, \tag{4.3.2}$$

where $\mathscr{A}'[l^m](\mathfrak{U}')$ is the group of sections of $\mathscr{A}'[l^m] \to \mathfrak{U}'$.

Proof. We first construct by induction on $n \in \mathbb{N}$ a sequence $\{q_n : \mathfrak{U}_n \to \mathfrak{U}\}_{n \in \mathbb{N}}$ of finite étale morphisms with $\mathfrak{U}_n$ irreducible such that $\mathscr{A}_n[l^n] = \coprod_{\tau \in \mathscr{A}_n[l^n](\mathfrak{U}_n)} \tau(\mathfrak{U}_n)$, where $f_n : \mathscr{A}_n \to \mathfrak{U}_n$ denotes the base-change of f by q_n. Since there does not exist a residue field with characteristic l on $\mathfrak{U}$, the morphism $\mathscr{A}[l^n] \to \mathfrak{U}$ is finite and étale for any $n \in \mathbb{N}$. It follows that there exists a finite étale morphism $q_1 : \mathfrak{U}_1 \to \mathfrak{U}$ with $\mathfrak{U}_1$ irreducible such that $\mathscr{A}_1[l] = \coprod_{\tau \in \mathscr{A}_1[l](\mathfrak{U}_1)} \tau(\mathfrak{U}_1)$. Let m be a natural number, and suppose that we have constructed a sequence $\{q_n : \mathfrak{U}_n \to \mathfrak{U}\}_{n=1,\dots,m}$ satisfying the condition. Then by the same argument as above, there exists a finite étale morphism $r : \mathfrak{U}_{m+1} \to \mathfrak{U}_m$ with $\mathfrak{U}_{m+1}$ irreducible such that $\mathscr{A}_{m+1}[l^{m+1}] = \coprod_{\tau \in \mathscr{A}_{m+1}[l^{m+1}](\mathfrak{U}_{m+1})} \tau(\mathfrak{U}_{m+1})$. Let $q_{m+1} : \mathfrak{U}_{m+1} \to \mathfrak{U}$ be the composite $q_m \circ r$. Then q_{m+1} satisfies the required condition. Thus we obtain $\{q_n : \mathfrak{U}_n \to \mathfrak{U}\}_{n \in \mathbb{N}}$ by induction.

Let $\mathscr{D}_n$ be the pullback of $\mathscr{D}$ by $\mathscr{A}_n \to \mathscr{A}$, and put

$$\mathscr{F}_n := \bigcap_{\tau \in \mathscr{A}_n[l^n](\mathfrak{U}_n)} \operatorname{Supp}\left(T_\tau^*(\mathscr{D}_n) + T_{-\tau}^*(\mathscr{D}_n)\right).$$

Then $\mathfrak{F}_n := q_n(f_n(\mathscr{F}_n))$ is a closed subset of $\mathfrak{U}$. Noting that there exists a natural map $\mathscr{A}_n[l^n](\mathfrak{U}_n) \hookrightarrow \mathscr{A}_{n+1}[l^{n+1}](\mathfrak{U}_{n+1})$, we see that $\mathfrak{F}_n \supset \mathfrak{F}_{n+1}$ for any $n \in \mathbb{N}$. Since $\mathfrak{U}$ is noetherian, there exists an $m \in \mathbb{N}$ such that for any $n \geq m$, $\mathfrak{F}_n = \mathfrak{F}_m$ holds.

It then suffices to show $\mathfrak{F}_m = \emptyset$ for this lemma; Indeed, if this holds, letting $\mathfrak{U}' \to \mathfrak{U}$ be the morphism $q_m : \mathfrak{U}_m \to \mathfrak{U}$, we then see that m and this $\mathfrak{U}' \to \mathfrak{U}$ satisfy the required condition in the lemma. We prove $\mathfrak{F}_m = \emptyset$ by contradiction. Suppose that $\mathfrak{F}_m \neq \emptyset$. Then there exists a geometric point $\overline{u} \in \mathfrak{F}_m \subset \mathfrak{U}$. Since $\mathscr{D}$ is an effective Cartier divisor flat over $\mathfrak{U}$, $\mathscr{D} \cap f^{-1}(\overline{u})$ is an effective divisor on the abelian variety $f^{-1}(\overline{u})$. By Lemma 4.2, there exists an $m_0 \in \mathbb{N}$ with $m_0 \geq m$ such that

$$\bigcap_{\tau \in f^{-1}(\overline{u})[l^{m_0}]} \operatorname{Supp}\left(T_\tau^*\left(\mathscr{D} \cap f^{-1}(\overline{u})\right) + T_{-\tau}^*\left(\mathscr{D} \cap f^{-1}(\overline{u})\right)\right) = \emptyset.$$

For any $\overline{u}_{m_0} \in q_{m_0}^{-1}(\overline{u})$, remark that the natural morphism $\mathscr{A}_{m_0} \to \mathscr{A}$ restricts to an isomorphism $f_{m_0}^{-1}(\overline{u}_{m_0}) \cong f^{-1}(\overline{u})$, and via that isomorphism, we have $\mathscr{D}_{m_0} \cap f_{m_0}^{-1}(\overline{u}_{m_0}) = \mathscr{D} \cap f^{-1}(\overline{u})$. Then we have

$$\bigcap_{\tau \in f_{m_0}^{-1}(\overline{u}_{m_0})[l^{m_0}]} \operatorname{Supp}\left(T_\tau^*\left(\mathscr{D}_{m_0} \cap f_{m_0}^{-1}(\overline{u}_{m_0})\right) + T_{-\tau}^*\left(\mathscr{D}_{m_0} \cap f_{m_0}^{-1}(\overline{u}_{m_0})\right)\right) = \emptyset. \tag{4.3.3}$$

Since $\mathscr{A}_{m_0}[l^{m_0}] = \coprod_{\tau \in \mathscr{A}_{m_0}[l^{m_0}](\mathfrak{U}_{m_0})} \tau(\mathfrak{U}_{m_0})$ and the characteristic of the residue field of $\overline{u}_{m_0}$ does not equal l, the natural map $\mathscr{A}_{m_0}[l^{m_0}](\mathfrak{U}_{m_0}) \to f_{m_0}^{-1}(\overline{u}_{m_0})[l^{m_0}]$

given by restriction is an isomorphism. It follows that

$$\bigcap_{\tau \in f_{m_0}^{-1}(\overline{u}_{m_0})[l^{m_0}]} \operatorname{Supp}\left(T_\tau^*\left(\mathscr{D}_{m_0} \cap f_{m_0}^{-1}(\overline{u}_{m_0})\right) + T_{-\tau}^*\left(\mathscr{D}_{m_0} \cap f_{m_0}^{-1}(\overline{u}_{m_0})\right)\right)$$
$$= \bigcap_{\tilde{\tau} \in \mathscr{A}_{m_0}[l^{m_0}](\mathfrak{U}_{m_0})} \operatorname{Supp}\left(T_{\tilde{\tau}}^*(\mathscr{D}_{m_0}) + T_{-\tilde{\tau}}^*(\mathscr{D}_{m_0})\right) \cap f_{m_0}^{-1}(\overline{u}_{m_0}).$$

Therefore, (4.3.3) shows that

$$\bigcap_{\tilde{\tau} \in \mathscr{A}_{m_0}[l^{m_0}](\mathfrak{U}_{m_0})} \operatorname{Supp}\left(T_{\tilde{\tau}}^*(\mathscr{D}_{m_0}) + T_{-\tilde{\tau}}^*(\mathscr{D}_{m_0})\right) \cap f_{m_0}^{-1}(\overline{u}_{m_0}) = \emptyset$$

for any $\overline{u}_{m_0} \in q_{m_0}^{-1}(\overline{u})$. On the other hand, since $\overline{u} \in \mathfrak{F}_m = \mathfrak{F}_{m_0}$, there exists an $\overline{u}'_{m_0} \in q_{m_0}^{-1}(\overline{u})$ such that

$$\bigcap_{\tilde{\tau} \in \mathscr{A}_{m_0}[l^{m_0}](\mathfrak{U}_{m_0})} \operatorname{Supp}\left(T_{\tilde{\tau}}^*(\mathscr{D}_{m_0}) + T_{-\tilde{\tau}}^*(\mathscr{D}_{m_0})\right) \cap f_{m_0}^{-1}(\overline{u}'_{m_0}) = \mathscr{F}_{m_0} \cap f_{m_0}^{-1}(\overline{u}'_{m_0}) \neq \emptyset.$$

However, that is a contradiction. Thus we obtain $\mathfrak{F}_m = \emptyset$, which completes the proof of the lemma. $\square$

Now we show the following base-point freeness on a model.

Proposition 4.4. *Let K' be a finite extension of K, let $\mathfrak{B}'$ be the normalization of $\mathfrak{B}$ in K', let $\mathfrak{U}$ be an open subset of $\mathfrak{B}'$ with* $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$, *and let $f : \mathscr{A} \to \mathfrak{U}$ be an abelian scheme with zero-section 0_f. Let $\mathscr{D}$ be an effective Cartier divisor on $\mathscr{A}$ that is flat over $\mathfrak{U}$. Suppose that the restriction of $\mathcal{O}_{\mathscr{A}}(\mathscr{D})$ to the generic fiber of f is an even line bundle and that $0_f^*\left(\mathcal{O}_{\mathscr{A}}(\mathscr{D})\right) \cong \mathcal{O}_{\mathfrak{U}}$. Then there exists a finite étale morphism $\mathfrak{U}_1 \to \mathfrak{U}$ with $\mathfrak{U}_1$ irreducible and normal that satisfies the following condition: Let $f_1 : \mathscr{A}_1 \to \mathfrak{U}_1$ and $\mathscr{D}_1$ be the base-change of f and $\mathscr{D}$ by this $\mathfrak{U}_1 \to \mathfrak{U}$, respectively; then there exists a finite dimensional k-linear subspace $\mathscr{V}$ of $H^0\left(\mathscr{A}_1, \mathcal{O}_{\mathscr{A}_1}(2\mathscr{D}_1)\right)$ such that $\mathscr{V}$ is base-point free, that is, the evaluation homomorphism $\mathscr{V} \otimes_k \mathcal{O}_{\mathscr{A}_1} \to \mathcal{O}_{\mathscr{A}_1}(2\mathscr{D}_1)$ is surjective.*

Proof. Let l be a prime number with $l \neq \operatorname{char}(k)$. By Lemma 4.3, there exist a natural number m and a finite étale morphism $\mathfrak{U}_1 \to \mathfrak{U}$ with $\mathfrak{U}_1$ irreducible such that

$$\bigcap_{\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)} \operatorname{Supp}\left(T_{\tilde{\tau}}^*(\mathscr{D}_1) + T_{-\tilde{\tau}}^*(\mathscr{D}_1)\right) = \emptyset, \tag{4.4.4}$$

where $f_1 : \mathscr{A}_1 \to \mathfrak{U}_1$ and $\mathscr{D}_1$ are the base-change of f and $\mathscr{D}$ by this $\mathfrak{U}_1 \to \mathfrak{U}$, respectively. Note that $\mathfrak{U}_1$ is normal as well as $\mathfrak{U}$. Indeed, since the fiber of $\mathfrak{U}_1 \to \mathfrak{U}$ over any point of $\mathfrak{U}$ is a finite reduced scheme, that follows from [11, Corollary of Theorem 23.9]. Let A be the geometric generic fiber of f_1, and let D be the restriction of $\mathscr{D}_1$ to A. Since $\operatorname{char}(k) \neq l$, $\bigcup_{n \in \mathbb{N}} A[l^n]$ is dense in A, and hence there exist an $n_0 \in \mathbb{N}$ and $\sigma \in A[l^{n_0}]$ such that

$$\sigma \notin \bigcup_{\tau \in A[l^m]} \operatorname{Supp}\left(T_\tau^*(D)\right). \tag{4.4.5}$$

Since $\operatorname{char}(k) \neq l$, there exists a finite étale morphism $\mathfrak{U}'_1 \to \mathfrak{U}_1$ with $\mathfrak{U}'_1$ irreducible such that σ extends to a section $\tilde{\sigma}$ of the base-change $\mathscr{A}'_1 \to \mathfrak{U}'_1$ of f_1 by this $\mathfrak{U}'_1 \to \mathfrak{U}_1$. Here, the function field of $\mathfrak{U}'_1$ is regarded as a subfield of $\overline{K}$. Replacing $\mathfrak{U}_1$ with $\mathfrak{U}'_1$, we may and do assume $\mathfrak{U}'_1 = \mathfrak{U}_1$.

We note by (4.4.5) that for any $\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)$, $\tilde{\sigma}(\mathfrak{U}_1) \nsubseteq \operatorname{Supp}(T^*_{\tilde{\tau}}(\mathscr{D}_1))$ holds, and hence the effective Cartier divisor $\tilde{\sigma}^*\left(T^*_{\tilde{\tau}}(\mathscr{D}_1)\right)$ is well-defined.

Claim 4.4.6. Let $\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)$. Then the Cartier divisor $\tilde{\sigma}^*\left(T^*_{\tilde{\tau}}(\mathscr{D}_1)\right)$ on $\mathfrak{U}_1$ is trivial.

Proof. Remark that $\tilde{\sigma}^*\left(T^*_{\tilde{\tau}}(\mathscr{D}_1)\right) = (\tilde{\sigma}+\tilde{\tau})^*(\mathscr{D}_1)$ as Cartier divisors. Since $(\tilde{\sigma}+\tilde{\tau})^*(\mathscr{D}_1) = (f_1)_*\left(\mathscr{D}_1 \cap (\tilde{\sigma}+\tilde{\tau})(\mathfrak{U}_1)\right)$ as cycles on $\mathfrak{U}_1$, we then find

$$\tilde{\sigma}^*\left(T^*_{\tilde{\tau}}(\mathscr{D}_1)\right) = (f_1)_*\left(\mathscr{D}_1 \cap (\tilde{\sigma}+\tilde{\tau})(\mathfrak{U}_1)\right).$$

Let K_1 be the function field of $\mathfrak{U}_1$. Note that K_1 is a finite extension of K'. Let $\mathfrak{B}_1 \to \mathfrak{B}'$ be the normalization of $\mathfrak{B}'$ in K_1. Since $\mathfrak{U}_1$ is normal and $\mathfrak{U}_1 \to \mathfrak{U}$ is finite, we have $\mathfrak{U}_1 \subset \mathfrak{B}_1$; and since $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$ and $\mathfrak{U}_1 \to \mathfrak{U}$ is finite, we have $\operatorname{codim}(\mathfrak{B}_1 \setminus \mathfrak{U}_1, \mathfrak{B}_1) \geq 2$.

Note that A is nowhere degenerate by Remark 2.4. Let $\mathcal{H}_1$ be the pullback of the line bundle $\mathcal{H}$ on $\mathfrak{B}$ by the composite $\mathfrak{B}_1 \to \mathfrak{B}' \to \mathfrak{B}$. Take a Weil divisor $\mathfrak{D}_1$ on $\mathfrak{B}_1$ with $\mathfrak{D}_1 \cap \mathfrak{U}_1 = (f_1)_*\left(\mathscr{D}_1 \cap (\tilde{\sigma}+\tilde{\tau})(\mathfrak{U}_1)\right)$. Then we note

$$[\mathfrak{D}_1 \cap \mathfrak{U}_1] = (f_1)_*\left(c_1(\mathcal{O}_{\mathscr{A}_1}(\mathscr{D}_1)) \cdot [(\tilde{\sigma}+\tilde{\tau})(\mathfrak{U}_1)]\right)$$

as cycle classes, in particular. Since $L := \mathcal{O}_{\mathscr{A}_1}(\mathscr{D}_1)|_A$ is even and $0^*_{f_1}\left(\mathcal{O}_{\mathscr{A}_1}(\mathscr{D}_1)\right) \cong \mathcal{O}_{\mathfrak{U}_1}$ by assumption, where 0_{f_1} is the zero-section of f_1, it follows from Lemma 2.5 that

$$\widehat{h}_L(\sigma+\tau) = \frac{\deg_{\mathcal{H}_1}[\mathfrak{D}_1]}{[K_1 : K]}.$$

Since $\sigma+\tau$ is a torsion point, it follows from the above equality that $\deg_{\mathcal{H}_1}[\mathfrak{D}_1] = 0$. Since $\mathfrak{D}_1$ is effective and $\mathcal{H}_1$ is ample, this means that $\mathfrak{D}_1$ is trivial. Therefore

$$\tilde{\sigma}^*\left(T^*_{\tilde{\tau}}(\mathscr{D}_1)\right) = (f_1)_*\left(\mathscr{D}_1 \cap (\tilde{\sigma}+\tilde{\tau})(\mathfrak{U}_1)\right) = \mathfrak{D}_1 \cap \mathfrak{U}_1$$

is trivial. Thus the claim holds. □

Let us prove that for any $\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)$, we have $T^*_{\tilde{\tau}}(\mathscr{D}_1) + T^*_{-\tilde{\tau}}(\mathscr{D}_1) \sim 2\mathscr{D}_1$. Take any $\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)$. Put

$$\mathscr{N} := \mathcal{O}_{\mathscr{A}_1}\left(T^*_{\tilde{\tau}}(\mathscr{D}_1) + T^*_{-\tilde{\tau}}(\mathscr{D}_1) - 2\mathscr{D}_1\right).$$

We then have $\mathscr{N}|_A = \mathcal{O}_A\left(T^*_\tau(D) + T^*_{-\tau}(D) - 2D\right)$, where τ is the restriction of $\tilde{\tau}$ to A. By the theorem of the square (cf. [16, II 6 Corollary 4]), we have $\mathscr{N}|_A \cong \mathcal{O}_A$. This means that there exists a line bundle $\mathcal{M}$ on $\mathfrak{U}_1$ such that $\mathscr{N} = f_1^*(\mathcal{M})$; indeed, since f_1 has irreducible and reduced fibers, if we set $\mathcal{M} := 0^*_{f_1}(\mathscr{N})$, then $\mathscr{N} = f_1^*(\mathcal{M})$ (cf. the description of the Picard functor in the first paragraph of [17, Chapter 0, Section 5, d)]). By $0^*_{f_1}\left(\mathcal{O}_{\mathscr{A}_1}(\mathscr{D}_1)\right) \cong \mathcal{O}_{\mathfrak{U}_1}$ and Claim 4.4.6, we see that $\sigma^*(\mathscr{N})$ is trivial. Hence $\mathcal{M} = \sigma^*(\mathscr{N}) \cong \mathcal{O}_{\mathfrak{U}_1}$, and thus $\mathscr{N} = f_1^*(\mathcal{M})$ is trivial. This shows that $T^*_{\tilde{\tau}}(\mathscr{D}_1) + T^*_{-\tilde{\tau}}(\mathscr{D}_1) \sim 2\mathscr{D}_1$ for any $\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)$.

Therefore, for each $\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)$, there exists a section $s_{\tilde{\tau}} \in H^0\left(\mathscr{A}_1, \mathcal{O}_{\mathscr{A}_1}(2\mathscr{D}_1)\right)$ such that $\operatorname{div}(s_{\tilde{\tau}}) = T^*_{\tilde{\tau}}(\mathscr{D}_1) + T^*_{-\tilde{\tau}}(\mathscr{D}_1)$. Let $\mathscr{V}$ be the k-vector subspace of $H^0(\mathscr{A}_1, \mathcal{O}_{\mathscr{A}_1}(2\mathscr{D}_1))$ spanned by $\{s_{\tilde{\tau}}\}_{\tilde{\tau} \in \mathscr{A}_1[l^m](\mathfrak{U}_1)}$. Then by (4.4.4), $\mathscr{V}$ does not have base-points. Thus we obtain the proposition. □

4.3. **Positivity of the canonical height of an ample divisor.** The purpose of this subsection is to show Proposition 4.6, the positivity of the canonical height of an ample divisor. To do that, we show the following lemma, which will be the key to the proof of the proposition.

Lemma 4.5. *Let A be a nowhere degenerate abelian variety over $\overline{K}$. Let K', $\mathfrak{B}'$, $\mathfrak{U}$, $f : \mathscr{A} \to \mathfrak{U}$, and 0_f be as in Proposition* 4.4. *Assume that f is a model of A; that is, f has geometric generic fiber A. Let $\mathscr{D}$ be an effective Cartier divisor on $\mathscr{A}$ that is flat over any point of $\mathfrak{U}$ of codimension* 1. *Let D be the restriction of $\mathscr{D}$ to A. Assume that $\mathcal{O}_A(D)$ is even and ample. Further, suppose that $0_f^*\left(\mathcal{O}_{\mathscr{A}}(\mathscr{D})\right) \cong \mathcal{O}_{\mathfrak{U}}$. Then A has non-trivial $\overline{K}/k$-trace.*

Proof. Replacing $\mathfrak{U}$ with its open subset whose complement in $\mathfrak{U}$ has codimension at least 2 if necessary, we may and do assume that $\mathscr{D}$ is flat over $\mathfrak{U}$. Since $\mathcal{O}_A(D)$ is even and $0_f^*\left(\mathcal{O}_{\mathscr{A}}(\mathscr{D})\right) \cong \mathcal{O}_{\mathfrak{U}}$, Proposition 4.4 gives us a finite surjective morphism $\mathfrak{U}_1 \to \mathfrak{U}$ with $\mathfrak{U}_1$ irreducible and normal such that, if $f_1 : \mathscr{A}_1 \to \mathfrak{U}_1$ and $\mathscr{D}_1$ are the base-change of f and $\mathscr{D}$ by the morphism $\mathfrak{U}_1 \to \mathfrak{U}$, respectively, then there exists a finite dimensional linear subspace $\mathscr{V}$ of $H^0\left(\mathscr{A}_1, \mathcal{O}_{\mathscr{A}_1}(2\mathscr{D}_1)\right)$ such that the evaluation homomorphism $\mathscr{V} \otimes_k \mathcal{O}_{\mathscr{A}_1} \to \mathcal{O}_{\mathscr{A}_1}(2\mathscr{D}_1)$ is surjective. Note that there exists a normal projective variety $\mathfrak{B}_1$ which contains $\mathfrak{U}_1$ with $\operatorname{codim}\left(\mathfrak{B}_1 \setminus \mathfrak{U}_1, \mathfrak{B}_1\right) \geq 2$. Indeed, it suffices to let $\mathfrak{B}_1$ be the normalization of $\mathfrak{B}'$ in the function field of $\mathfrak{U}_1$; see the proof of Claim 4.4.6. Thus, replacing $f : \mathscr{A} \to \mathfrak{U}$ and $\mathscr{D}$ with $f_1 : \mathscr{A}_1 \to \mathfrak{U}_1$ and $\mathscr{D}_1$, respectively, we may and do assume that there exists a finite dimensional linear subspace $\mathscr{V}$ of $H^0\left(\mathscr{A}, \mathcal{O}_{\mathscr{A}}(2\mathscr{D})\right)$ such that the evaluation homomorphism $\mathscr{V} \otimes_k \mathcal{O}_{\mathscr{A}} \to \mathcal{O}_{\mathscr{A}}(2\mathscr{D})$ is surjective.

To ease notation, we put $\mathscr{L} := \mathcal{O}_{\mathscr{A}}(2\mathscr{D})$. Note that $0_f^*(\mathscr{L}) = 0_f^*\left(\mathcal{O}_{\mathscr{A}}(\mathscr{D})\right)^{\otimes 2} \cong \mathcal{O}_{\mathfrak{U}}$. Let $\varphi : \mathscr{A} \to \mathbb{P}_k^N$ be the morphism associated to the evaluation homomorphism $\mathscr{V} \otimes_k \mathcal{O}_{\mathscr{A}} \to \mathscr{L}$, where $N := \dim(\mathscr{V}) - 1$, and let $\mathcal{O}(1)$ denote the tautological line bundle on $\mathbb{P}_k^N$. Take any $\sigma \in A\left(\overline{K}\right)$, and let Δ_σ be the closure of σ in $\mathscr{A}$. Recall that $\mathcal{H}$ is an ample line bundle on $\mathfrak{B}$, and let $\mathcal{H}'$ be the pullback of $\mathcal{H}$ by the finite morphism $\mathfrak{B}' \to \mathfrak{B}$. Let m be a positive integer such that $(\mathcal{H}')^{\otimes m}$ is very ample. First, we apply Proposition 3.5 (1) to $\mathfrak{B}'$, $\mathfrak{U}$, and $(\mathcal{H}')^{\otimes m}$. Then there exists a dense open subset $V(k) \subset \left|(\mathcal{H}')^{\otimes m}\right|^{b-1}$ such that for any $\underline{D} = (D_1, \ldots, D_{b-1}) \in V(k)$, the intersection $D_1 \cap \cdots \cap D_{b-1}$ is a connected non-singular projective curve contained in $\mathfrak{U}$, where we recall that $b := \dim(\mathfrak{B}) = \dim(\mathfrak{B}')$.

We put $L := \mathscr{L}|_A = \mathcal{O}_A(2D)$. Remark that it is even and ample, since so is $\mathcal{O}_A(D)$ by the assumption.

Claim 4.5.7. Take any $\underline{D} = (D_1, \ldots, D_{b-1}) \in V(k)$, and set $C := D_1 \cap \cdots \cap D_{b-1}$. Then

$$\left(m^{b-1} \deg\left(f|_{\Delta_\sigma}\right)[K' : K]\right) \widehat{h}_L(\sigma) = \deg\left(c_1\left(\mathcal{O}(1)\right) \cdot \varphi_*\left(\left[\Delta_\sigma \cap f^{-1}(C)\right]\right)\right).$$

Furthermore, $\widehat{h}_L(\sigma) = 0$ if and only if $\dim(\varphi(\Delta_\sigma \cap f^{-1}(C))) = 0$.

Proof. By Nagata's embedding theorem (cf. [26, Theorem 5.7]), there exists a proper morphism $\bar{f} : \bar{\mathscr{A}} \to \mathfrak{B}'$ such that the restriction $\bar{f}^{-1}(\mathfrak{U}) \to \mathfrak{U}$ coincides with $f : \mathscr{A} \to \mathfrak{U}$. Furthermore, we can take $\bar{f}$ in such a way that $\varphi : \mathscr{A} \to \mathbb{P}_k^N$ extends to a morphism $\bar{\varphi} : \bar{\mathscr{A}} \to \mathbb{P}_k^N$. Indeed, this condition is satisfied if we replace $\bar{\mathscr{A}}$ with the graph of the rational map from $\bar{\mathscr{A}}$ to $\mathbb{P}_k^N$ given by φ. Set $\bar{\mathscr{L}} := \bar{\varphi}^*\left(\mathcal{O}(1)\right)$. Remark that $\bar{\mathscr{L}}|_{\mathscr{A}} = \mathscr{L}$. Let $\bar{\Delta}_\sigma$ be the closure of σ in $\bar{\mathscr{A}}$.

From the definition of C, we see that

$$m^{b-1}c_1\left(\mathcal{H}'\right)^{b-1}\cdot[\mathfrak{B}'] = c_1\left((\mathcal{H}')^{\otimes m}\right)^{b-1}\cdot[\mathfrak{B}'] = [C]$$

as 1-cycle classes on $\mathfrak{B}'$. Since $\bar{f}$ is proper and flat over C and since $C \subset \mathfrak{U}$, it follows that

$$m^{b-1}c_1\left(\bar{f}^*(\mathcal{H}')\right)^{b-1}\cdot\left[\bar{\Delta}_\sigma\right] = \left[f^{-1}(C)\cap\Delta_\sigma\right]$$

as 1-cycle classes on $\bar{\mathscr{A}}$. Therefore,

(4.5.8)
$$m^{b-1}c_1\left(\bar{\varphi}^*\left(\mathcal{O}(1)\right)\right)\cdot c_1(\bar{f}^*(\mathcal{H}'))^{\cdot(b-1)}\cdot\left[\bar{\Delta}_\sigma\right] = c_1\left(\bar{\varphi}^*\left(\mathcal{O}(1)\right)\right)\cdot\left[\Delta_\sigma\cap f^{-1}(C)\right]$$

as 0-cycle classes on $\bar{\mathscr{A}}$. By the projection formula, the degree of the left-hand side of (4.5.8) equals

$$m^{b-1}\deg_{\mathcal{H}'}\bar{f}_*\left(c_1\left(\bar{\varphi}^*\left(\mathcal{O}(1)\right)\right)\cdot\left[\bar{\Delta}_\sigma\right]\right).$$

Since $\bar{\varphi}^*\left(\mathcal{O}(1)\right)|_A = L$ and $0_f^*\left(\bar{\varphi}^*\left(\mathcal{O}(1)\right)\right) = 0_f^*\left(\mathscr{L}\right) \cong \mathcal{O}_{\mathfrak{U}}$, that equals

$$\left(m^{b-1}\deg\left(f|_{\Delta_\sigma}\right)[K':K]\right)\widehat{h}_L(\sigma)$$

by Lemma 2.5. On the other hand, the right-hand side of (4.5.8) equals

$$c_1\left(\mathcal{O}(1)\right)\cdot\varphi_*\left(\left[\Delta_\sigma\cap f^{-1}(C)\right]\right).$$

Thus we obtain

$$\left(m^{b-1}\deg\left(f|_{\Delta_\sigma}\right)[K':K]\right)\widehat{h}_L(\sigma) = \deg\left(c_1\left(\mathcal{O}(1)\right)\cdot\varphi_*\left(\left[\Delta_\sigma\cap f^{-1}(C)\right]\right)\right),$$

as required.

The last assertion of the claim follows from the fact that $\mathcal{O}(1)$ is ample and $\varphi_*\left(\left[\Delta_\sigma\cap f^{-1}(C)\right]\right)$ is effective. □

Claim 4.5.10. We have $\widehat{h}_L(\sigma) = 0$ if and only if $\varphi(\Delta_\sigma)$ consists of a single point.

Proof. First, we show the "if" part. Suppose that $\varphi(\Delta_\sigma)$ is a singleton. Recall that $V(k)$ is the dense open subset of $|(\mathcal{H}')^{\otimes m}|^{b-1}$ taken just above Claim 4.5.7. Since $V(k) \neq \emptyset$, we take a $(D_1,\ldots,D_{b-1}) \in V(k)$ and set $C := D_1\cap\cdots\cap D_{b-1}$. Then C is an irreducible curve contained in $\mathfrak{U}$. Since $\varphi(f^{-1}(C)\cap\Delta_\sigma) \subset \varphi(\Delta_\sigma)$, we have $\dim(\varphi(f^{-1}(C)\cap\Delta_\sigma)) = 0$ by the assumption. By Claim 4.5.7, we conclude $\widehat{h}_L(\sigma) = 0$.

Let us prove the other implication. By Proposition 3.5 (2), for a general $x \in \mathfrak{U}(k)$, the set $V_x(k) := V(k)\cap|(\mathcal{H}')^{\otimes m}|_x^{b-1}$ is dense in $|(\mathcal{H}')^{\otimes m}|_x^{b-1}$, where

$$\left|(\mathcal{H}')^{\otimes m}\right|_x^{b-1} := \left\{(D_1,\ldots,D_{b-1}) \in \left|(\mathcal{H}')^{\otimes m}\right|^{b-1}\,\middle|\, x \in D_1\cap\cdots\cap D_{b-1}\right\}.$$

Since $f|_{\Delta_\sigma} : \Delta_\sigma \to \mathfrak{U}$ is generically finite and generically flat, we take such an x so that f is finite and flat over an open neighborhood of x.

We set

$$\mathscr{S}(x) := \{D_1\cap\cdots\cap D_{b-1} \mid (D_1,\ldots,D_{b-1}) \in V_x(k)\}.$$

Let y be a point of Δ_σ with $f(y) = x$, and set $\mathscr{T}_\sigma(y)$ to be

$$\{C' \mid C' \text{ is an irreducible component of } f^{-1}(C)\cap\Delta_\sigma \text{ with } y \in C' \text{ for some } C \in \mathscr{S}(x)\}.$$

Since $f|_{\Delta_\sigma} : \Delta_\sigma \to \mathfrak{U}$ is surjective and is finite flat over x and since any $C \in \mathscr{S}(x)$ is irreducible, we find that $\{f(C') \mid C' \in \mathscr{T}_\sigma(y)\} = \mathscr{S}(x)$. Therefore

$$f\left(\bigcup_{C' \in \mathscr{T}_\sigma(y)} C'\right) = \bigcup_{C \in \mathscr{S}(x)} C.$$

By Proposition 3.5 (3), this subset is dense in $\mathfrak{B}'$. Since Δ_σ is irreducible and $f|_{\Delta_\sigma}$ is generically finite, it follows that $\bigcup_{C' \in \mathscr{T}_\sigma(y)} C'$ is dense in Δ_σ.

Suppose $\widehat{h}_L(\sigma) = 0$. By Claim 4.5.7, we have $\dim(\varphi(\Delta_\sigma \cap f^{-1}(C))) = 0$ for any $C \in \mathscr{S}(x)$. This means that for any $C' \in \mathscr{T}_\sigma(y)$, we have $\varphi(C') = \varphi(y)$. It follows that $\varphi(\bigcup_{C' \in \mathscr{T}(y)} C') = \varphi(y)$. Since $\bigcup_{C' \in \mathscr{T}_\sigma(y)} C'$ is dense in Δ_σ, this concludes $\varphi(\Delta_\sigma) = \varphi(y)$, which is a singleton. This completes the proof of the claim. $\square$

Note that $\mathscr{L}|_{f^{-1}(u)}$ is ample for any $u \in \mathfrak{U}$. Indeed, since $\mathscr{L}$ is ample on the generic fiber of f, it is relatively ample over a dense open subset of $\mathfrak{U}$. By the flatness of f, we see that $\mathscr{L}|_{f^{-1}(u)}$ is big for any $u \in \mathfrak{U}$. Since ampleness is the same as bigness on an abelian variety, this means that $\mathscr{L}|_{f^{-1}(u)}$ is ample for any $u \in \mathfrak{U}$.

Claim 4.5.11. For any closed subvariety $C \subset \mathscr{A}$ of dimension 1 such that $\dim f(C) = 0$, we have $\dim(\varphi(C)) > 0$.

Proof. Since $\dim f(C) = 0$, we put $u := f(C)$, which is a closed point of $\mathfrak{U}$. Remark that C is a closed curve of the abelian variety $f^{-1}(u)$. Since $\mathscr{L}|_{f^{-1}(u)}$ is ample, we then have

$$0 < \deg(c_1(\mathscr{L}) \cdot [C]) = \deg\left(\varphi^*\left(c_1\left(\mathcal{O}(1)\right)\right) \cdot [C]\right) = \deg\left(c_1\left(\mathcal{O}(1)\right) \cdot \varphi_*([C])\right),$$

which shows $\dim(\varphi(C)) > 0$. $\square$

Put

$$\Gamma := \{\gamma \mid \gamma \text{ is an irreducible closed subset of } \mathscr{A} \text{ with } \dim(\gamma) = b \text{ and } f(\gamma) = \mathfrak{U}\}.$$

For any $\sigma \in A\left(\overline{K}\right)$, we find $\Delta_\sigma \in \Gamma$, and this assignment defines a map $A\left(\overline{K}\right) \to \Gamma$. Note that it is surjective. Further, put

$$\Gamma_0 := \{\gamma \in \Gamma \mid \dim(\varphi(\gamma)) = 0\}$$

and

$$A\left(\overline{K}\right)_0 := \left\{\sigma \in A\left(\overline{K}\right) \,\middle|\, \widehat{h}_L(\sigma) = 0\right\}.$$

Then by Claim 4.5.10, the map $A\left(\overline{K}\right) \to \Gamma$ induces a surjective map $\alpha : A\left(\overline{K}\right)_0 \to \Gamma_0$.

Since the scheme-theoretic image $\varphi(\mathscr{A})$ is an integral scheme, there exists a dense open subset $Z \subset \varphi(\mathscr{A})$ such that the restriction $\varphi^{-1}(Z) \to Z$ is flat. Remark that since $\mathscr{A}$ is irreducible, $\varphi^{-1}(z)$ is pure-dimensional for any $z \in Z$.

Claim 4.5.12. Let $z \in Z(k)$, and let Δ be an irreducible component of $\varphi^{-1}(z)$. Then $f|_\Delta : \Delta \to \mathfrak{U}$ is a finite surjective morphism.

Proof. First, we prove that $\dim(\Delta) \geq b$. Remark that for any $\sigma \in A\left(\overline{K}\right)_0$, $\alpha(\sigma) = \Delta_\sigma$ is a closed subset of $\mathscr{A}$. Since $A\left(\overline{K}\right)_0$ is dense in A, $\bigcup_{\sigma \in A\left(\overline{K}\right)_0} \alpha(\sigma)$ is dense in $\mathscr{A}$. It follows that there exists a $\sigma \in A\left(\overline{K}\right)_0$ such that the generic point of $\alpha(\sigma)$ is in $\varphi^{-1}(Z)$. Since $\dim \varphi(\alpha(\sigma)) = 0$, we then have $\varphi(\alpha(\sigma)) \in Z(k)$. Furthermore, since $\dim(\alpha(\sigma)) = b$, it follows that $\dim \varphi^{-1}(\varphi(\alpha(\sigma))) \geq b$. Since φ is flat over Z and $\varphi|_{\varphi^{-1}(Z)} : \varphi^{-1}(Z) \to Z$ is surjective, we find $\dim \varphi^{-1}(z) \geq b$, and thus $\dim(\Delta) \geq b$.

To conclude the claim, since $f|_{\varphi^{-1}(z)} : \varphi^{-1}(z) \to \mathfrak{U}$ is proper and any irreducible component of $\varphi^{-1}(z)$ has dimension at least b, it suffices to show that $f|_{\varphi^{-1}(z)}$ is quasi-finite. We prove this by contradiction. Suppose that $f|_{\varphi^{-1}(z)}$ is not quasi-finite. Then there exists an irreducible reduced closed curve $C \subset \varphi^{-1}(z)$ such that $f(C)$ consists of a single point. By Claim 4.5.11, we have $\dim \varphi(C) > 0$. On the other hand, we have $\varphi(C) \subset \varphi\left(\varphi^{-1}(z)\right) = \{z\}$. That is a contradiction. □

Now, we prove the lemma. Set $\Gamma_0^Z := \{\gamma \in \Gamma_0 \mid \varphi(\gamma) \in Z\}$. Then for any $z \in Z(k)$, Claim 4.5.12 tells us that any irreducible component of $\varphi^{-1}(z)$ is in Γ_0^Z. This means in particular that the map $\Gamma_0^Z \to Z(k)$ which assigns to each $\gamma \in \Gamma_0^Z$ the point $\varphi(\gamma)$ is surjective. Since $\alpha : A\left(\overline{K}\right)_0 \to \Gamma_0$ is surjective, we then find that

$$\#A\left(\overline{K}\right)_0 \geq \#\Gamma_0 \geq \#\Gamma_0^Z \geq \#Z(k). \tag{4.5.13}$$

We set $n := \dim(A) > 0$. Since Claim 4.5.11 shows that the restriction of φ to a closed fiber of f is a finite morphism, we note that $\dim(Z) = \dim \varphi(\mathscr{A}) \geq n$.

First, assume that k has uncountably infinite cardinality, and we prove that A has non-trivial $\overline{K}/k$-trace under this condition first. Let $\aleph_0$ denote the countably infinite cardinality. Then since $\dim(Z) > 0$, we have $\#Z(k) > \aleph_0$. It follows from inequality (4.5.13) that $\#A\left(\overline{K}\right)_0 > \aleph_0$. Since $\#A\left(\overline{K}\right)_{tor} = \aleph_0$, this means that $A\left(\overline{K}\right)$ has height 0 points other than torsion points. By [10, Chapter 6, Theorem 5.5], we conclude that A has non-trivial $\overline{K}/k$-trace in this case.

Finally, we consider the general case. Take an algebraically closed field extension k' of k such that $\#k' > \aleph_0$. Then $\mathfrak{B} \otimes_k k'$ is a normal projective variety over k'. Further, $\mathfrak{U} \otimes_k k'$, $f \otimes_k k' : \mathscr{A} \otimes_k k' \to \mathfrak{U} \otimes_k k'$, and $\mathscr{D} \otimes_k k'$ satisfy all the conditions of the lemma as a variety, a morphism, and a Cartier divisor over the constant field k', respectively. Let F be the function field of $\mathfrak{B} \otimes_k k'$. Since we know that the lemma holds if the constant field has cardinality greater than $\aleph_0$, it turns out that $A \otimes_{\overline{K}} \overline{F}$ has non-trivial $\overline{F}/k'$-trace. By [31, Lemma A.1], it follows that A has non-trivial $\overline{K}/k$-trace. Thus we obtain the lemma. □

Now we show the positivity assertion.

Proposition 4.6. *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace, and let L be an even ample line bundle on A. Further, let X be an effective ample divisor on A. Then we have $\widehat{h}_L(X) > 0$.*

Proof. There exists a torsion point $\tau \in A\left(\overline{K}\right)$ such that $0 \notin \mathrm{Supp}(T_\tau^*(X))$. Since being $\widehat{h}_L(X) > 0$ is equivalent to being $\widehat{h}_L(T_\tau^*(X)) > 0$, we may assume that $0 \notin X$. We put $D := X + [-1]^*(X)$, where "+" means the addition of the divisors. Then D is an even ample divisor with $0 \notin \mathrm{Supp}(D)$. Since whether $\widehat{h}_L(X) > 0$ does or does not depend on the choice of even ample line bundles L, we may assume that $L = \mathcal{O}_A(D)$. Furthermore, since $[-1]$ is an automorphism and $[-1]^*(L) = L$, we have $\widehat{h}_L(X) = \widehat{h}_L(D)/2$, so that it suffices to show that $\widehat{h}_L(D) > 0$.

By Proposition 2.2, there exist a finite extension K' of K, an open subset $\mathfrak{U}$ of $\mathfrak{B}'$ with $\mathrm{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$, where $\mathfrak{B}'$ is the normalization of $\mathfrak{B}$ in K', an abelian scheme $f : \mathscr{A} \to \mathfrak{U}$ with zero-section 0_f, and a line bundle $\mathscr{L}$ on $\mathscr{A}$ such that $(f, \mathscr{L})$ is a model of (A, L) over $\mathfrak{U}$ with $0_f^*(\mathscr{L}) \cong \mathcal{O}_{\mathfrak{U}}$. Replacing this K' with its finite extension if necessary, we may and do assume that D can be defined over the function field K'. Let $\mathscr{D}$ be the closure of D in $\mathscr{A}$. Since D can be defined over K', the restriction of $\mathscr{D}$ to A coincides with D. Note that $\mathscr{D}$ is flat over any codimension 1 point of $\mathfrak{U}$. Since A has trivial $\overline{K}/k$-trace, it follows from Lemma 4.5 that $0_f^*(\mathcal{O}_{\mathscr{A}}(\mathscr{D}))$ is a non-trivial line bundle on $\mathfrak{U}$. Since $0_f(\mathfrak{U}) \not\subseteq \mathrm{Supp}(\mathscr{D})$ by assumption, $0_f^*(\mathscr{D})$ is a non-trivial effective Cartier divisor on $\mathfrak{U}$.

By Nagata's embedding theorem (cf. [26, Theorem 5.9]), there exists a proper morphism $\bar{f} : \bar{\mathscr{A}} \to \mathfrak{B}'$ such that $\bar{f}$ equals f over $\mathfrak{U}$ and that the Cartier divisors $\mathscr{D}$ and $f^*(0_f^*(\mathscr{D}))$ on $\mathscr{A}$ extend to Cartier divisors $\bar{\mathscr{D}}$ and $\bar{\mathscr{E}}$ on $\bar{\mathscr{A}}$, respectively. Since $\mathscr{D}$ is flat over any point of $\mathfrak{U}$ of codimension 1 and since $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$, $\bar{\mathscr{D}}$ is flat over any point of $\mathfrak{B}'$ of codimension 1. We set $\bar{\mathscr{L}} := \mathcal{O}_{\bar{\mathscr{A}}}(\bar{\mathscr{D}})$, $\bar{\mathscr{N}} := \mathcal{O}_{\bar{\mathscr{A}}}(\bar{\mathscr{E}})$, and $\bar{\mathscr{L}}_0 := \bar{\mathscr{L}} \otimes \bar{\mathscr{N}}^{\otimes(-1)}$.

Let $\mathcal{H}'$ be the pullback of $\mathcal{H}$ by the finite morphism $\mathfrak{B}' \to \mathfrak{B}$, and let m' be an integer such that $\mathcal{H}'' := (\mathcal{H}')^{\otimes m'}$ is very ample. Put $n := \dim(A)$. Then, $\widehat{h}_L(D) > 0$ if and only if

$$\deg\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot c_1\left(\bar{\mathscr{N}}\right) \cdot \bar{f}^* c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \left[\bar{\mathscr{A}}\,\right]\right) > 0. \tag{4.6.14}$$

Indeed, since $\bar{\mathscr{L}}_0|_A = L$, $0_f^*\left(\bar{\mathscr{L}}_0\right) \cong \mathcal{O}_{\mathfrak{U}}$, and since $\bar{\mathscr{D}}$ is flat over any point of $\mathfrak{B}'$ of codimension 1, we have, by [31, Lemma 2.6],

$$\widehat{h}_L(D) = \frac{\deg\left(c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \bar{f}_*\left(c_1(\bar{\mathscr{L}}_0)^{\cdot n} \cdot [\bar{\mathscr{D}}]\right)\right)}{(m')^{b-1}[K' : K]}.$$

By Remark 2.6, we have

$$\deg\left(c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \bar{f}_*\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot(n+1)} \cdot \left[\bar{\mathscr{A}}\,\right]\right)\right) = 0.$$

Since $\mathcal{O}_{\bar{\mathscr{A}}}\left(\bar{\mathscr{D}}\right) = \bar{\mathscr{L}}_0 \otimes \bar{\mathscr{N}}$, it follows that

$$\begin{aligned}
&\deg\left(c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \bar{f}_*\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot [\bar{\mathscr{D}}]\right)\right) \\
&= \deg\left(c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \bar{f}_*\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot \left(c_1\left(\bar{\mathscr{L}}_0\right) + c_1\left(\bar{\mathscr{N}}\right)\right) \cdot \left[\bar{\mathscr{A}}\,\right]\right)\right) \\
&= \deg\left(c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \bar{f}_*\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot c_1\left(\bar{\mathscr{N}}\right) \cdot \left[\bar{\mathscr{A}}\,\right]\right)\right) \\
&= \deg\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot c_1\left(\bar{\mathscr{N}}\right) \cdot \bar{f}^* c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \left[\bar{\mathscr{A}}\,\right]\right).
\end{aligned}$$

Thus $\widehat{h}_L(D) > 0$ if and only if (4.6.14) holds.

Let us prove (4.6.14). Since $\operatorname{codim}(\mathfrak{B}' \setminus \mathfrak{U}, \mathfrak{B}') \geq 2$ and $\mathcal{H}''$ is a very ample line bundle on $\mathfrak{B}'$, there exists a proper curve C on $\mathfrak{B}'$ such that $C \subset \mathfrak{U}'$, C intersects with $0_f^*(\mathscr{D})$ properly, and $c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot [\mathfrak{B}'] = [C]$ as cycle classes on $\mathfrak{B}'$ (cf. Proposition 3.5). Set $Z := C \cap 0_f^*(\mathscr{D})$. Since $\mathcal{H}''$ is ample and $0_f^*(\mathscr{D})$ is a non-trivial effective Cartier divisor, we note that Z is a non-trivial effective 0-cycle, and hence $\deg[Z] = \operatorname{length}(\mathcal{O}_Z) > 0$. Since $\bar{f}$ is flat over $\mathfrak{U}$, we note $\bar{f}^* c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot [\bar{\mathscr{A}}\,] = [\bar{f}^{-1}(C)]$. Since $\bar{f}^{-1}(C) \subset \mathscr{A}$ and $\bar{\mathscr{N}}|_{\mathscr{A}} = f^*(\mathcal{O}_{\mathfrak{U}}(0_f^*(\mathscr{D})))$, we find that $c_1(\bar{\mathscr{N}}) \cdot [\bar{f}^{-1}(C)] = \bar{f}^*[0_f^*(\mathscr{D}) \cap C] = \bar{f}^*[Z]$. It follows that

$$\deg\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot c_1\left(\bar{\mathscr{N}}\right) \cdot \bar{f}^* c_1(\mathcal{H}'')^{\cdot(b-1)} \cdot \left[\bar{\mathscr{A}}\,\right]\right) = \deg\left(c_1\left(\bar{\mathscr{L}}_0\right)^{\cdot n} \cdot \bar{f}^*[Z]\right). \tag{4.6.15}$$

Note that since $\bar{f}$ is flat over $\mathfrak{U}$, for any $u, u' \in \mathfrak{U}$, we have

(4.6.16)
$$\deg\left(c_1\left(\bar{\mathscr{L}_0}\big|_{\bar{f}^{-1}(u)}\right)^{\cdot n}\cdot\left[\bar{f}^{-1}(u)\right]\right)=\deg\left(c_1\left(\bar{\mathscr{L}_0}\big|_{\bar{f}^{-1}(u')}\right)^{\cdot n}\cdot\left[\bar{f}^{-1}(u')\right]\right).$$

Here, suppose that $u\in\mathfrak{U}(k)$. Then the left-hand side of (4.6.16) equals $\deg(c_1(\bar{\mathscr{L}_0})^{\cdot n}\cdot[\bar{f}^{-1}(u)])$. On the other hand, suppose that u' is the generic point of $\mathfrak{U}$. Then the right-hand side of (4.6.16) equals $\deg(c_1(\bar{\mathscr{L}_0}|_A)^{\cdot n}\cdot[A])$. Thus we have

$$\deg\left(c_1\left(\bar{\mathscr{L}_0}\right)^{\cdot n}\cdot\left[\bar{f}^{-1}(u)\right]\right)=\deg\left(c_1\left(\bar{\mathscr{L}_0}\big|_A\right)^{\cdot n}\cdot[A]\right). \tag{4.6.17}$$

Since Z is a 0-cycle, this is a sum of closed points. Further, the support of Z is contained in $\mathfrak{U}$. Therefore, it follows from (4.6.17) that

$$\deg\left(c_1\left(\bar{\mathscr{L}_0}\right)^{\cdot n}\cdot f^*[Z]\right)=\deg[Z]\cdot\deg\left(c_1\left(\bar{\mathscr{L}_0}\big|_A\right)^{\cdot n}\cdot[A]\right). \tag{4.6.18}$$

Since $\bar{\mathscr{L}_0}|_A = L$ is ample, $\deg(c_1(\bar{\mathscr{L}_0}|_A)^{\cdot n}\cdot[A]) > 0$. Thus by (4.6.15) and (4.6.18), we obtain

$$\deg\left(c_1\left(\bar{\mathscr{L}_0}\right)^{\cdot n}\cdot c_1\left(\bar{\mathscr{N}}\right)\cdot\bar{f}^*c_1(\mathcal{H}'')^{\cdot(b-1)}\cdot\left[\bar{\mathscr{A}}\right]\right)>0.$$

This completes the proof of the proposition. $\square$

This is a small remark: The assumption that X is ample is made in Proposition 4.6, but in fact it can be removed; see Proposition 5.8 and Remark 5.9.

5. Non-density of small points

5.1. **Non-density of small points on closed subvarieties.** In this section, we focus on the non-density of small points on specified closed subvarieties of a given abelian variety: Suppose we are given an abelian variety A over $\overline{K}$; let X be a closed subvariety; then we consider the following assertion for X.

Conjecture 5.1. If X has dense small points, then X is a special subvariety.

We regard Conjecture 5.1 as a conjecture for X while we regard the geometric Bogomolov conjecture (Conjecture 1.5) as a conjecture for A. The geometric Bogomolov conjecture for A is equivalent to Conjecture 5.1 for all closed subvarieties X of A.

Remark 5.2. For a closed subvariety of dimension $\dim(A)$, Conjecture 5.1 holds trivially. For a closed subvariety of dimension 0, it is classically known that Conjecture 5.1 holds (cf. [30, Remark 7.4]). Therefore, the geometric Bogomolov conjecture holds for abelian varieties of dimension at most 1.

5.2. **Relative height.** In this subsection, we fix the following. Let A be a nowhere degenerate abelian variety over $\overline{K}$, and let L be an even ample line bundle on A. Let $\widetilde{Y}$ be a variety over k, and set $Y := \widetilde{Y}\otimes_k \overline{K}$. Let $p: Y\times A\to Y$ be the natural projection, which is an abelian scheme over Y. Let X be a closed subvariety of $Y\times A$ such that $p(X)=Y$.

In [31, Section 4.2], we defined an $\mathbb{R}$-valued function $\mathbf{h}^L_{X/Y}$ over some subset $\widetilde{Y}_{\mathrm{pd}}$ of $\widetilde{Y}$, called the relative height function. Roughly speaking, it is a function that assigns to each $\tilde{y}\in\widetilde{Y}_{\mathrm{pd}}$ the canonical height of the fiber of $p|_X$ over a geometric point of Y which corresponds to $\tilde{y}$ in a canonical way. We defined indeed $\mathbf{h}^L_{X/Y}$ not only for closed points of $\widetilde{Y}$ but also for general points. However, in the later argument in this paper, it will be enough for us to consider $\mathbf{h}^L_{X/Y}$ only over a set of closed points.

Therefore, we recall how $\mathbf{h}^L_{X/Y}$ is given for closed points. Let $\tilde{y}$ be a point in $\widetilde{Y}(k)$. Note that $\tilde{y}$ is regarded as a point of $Y\left(\overline{K}\right)$ naturally. Indeed, since $\tilde{y}$ is a morphism $\operatorname{Spec}(k) \to \widetilde{Y}$, taking the fiber product with $\operatorname{Spec}\left(\overline{K}\right)$ over $\operatorname{Spec}(k)$, we obtain $\operatorname{Spec}\left(\overline{K}\right) \to \widetilde{Y} \otimes_k \overline{K} = Y$, which is a point in $Y\left(\overline{K}\right)$. Thus we have $\widetilde{Y}(k) \hookrightarrow Y\left(\overline{K}\right)$. We denote by $\tilde{y}_{\overline{K}}$ the point in $Y\left(\overline{K}\right)$ corresponding to $\tilde{y}$ via this inclusion.

The fiber $p^{-1}\left(\tilde{y}_{\overline{K}}\right)$ is an abelian variety; indeed, the second projection $Y \times A \to A$ restricts to an isomorphism $p^{-1}\left(\tilde{y}_{\overline{K}}\right) \cong A$. We set $X_{\tilde{y}_{\overline{K}}} := p|_X^{-1}\left(\tilde{y}_{\overline{K}}\right) = p^{-1}\left(\tilde{y}_{\overline{K}}\right) \cap X$, which is a closed subscheme of that abelian variety. If follows that if $X_{\tilde{y}_{\overline{K}}}$ has a pure dimension, then the canonical height $\widehat{h}_L\left(X_{\tilde{y}_{\overline{K}}}\right)$ of $X_{\tilde{y}_{\overline{K}}}$ with respect to L is defined. Thus we set

$$\widetilde{Y}_{\mathrm{pd}}(k) := \left\{\tilde{y} \in \widetilde{Y}(k) \;\middle|\; X_{\tilde{y}_{\overline{K}}} \text{ has pure dimension } \dim(X) - \dim(Y)\right\}, \tag{5.2.19}$$

and by definition, $\mathbf{h}^L_{X/Y}$ is given by

$$\mathbf{h}^L_{X/Y}(\tilde{y}) = \widehat{h}_L\left(X_{\tilde{y}_{\overline{K}}}\right) \tag{5.2.20}$$

for $\tilde{y} \in \widetilde{Y}_{\mathrm{pd}}(k)$.

Remark 5.3. In [31, Section 4.2], for any $\tilde{y} \in \widetilde{Y}$, we defined a geometric point $\overline{\tilde{y}_K}$ of Y, which coincides with the above $\tilde{y}_{\overline{K}}$ when $\tilde{y} \in \widetilde{Y}(k)$. Further, we defined $\widetilde{Y}_{\mathrm{pd}}$ to be the set of points $\tilde{y} \in \widetilde{Y}$ such that $X_{\overline{\tilde{y}_K}}$ has pure dimension $\dim(X) - \dim(Y)$, where $X_{\overline{\tilde{y}_K}}$ is the fiber of $p|_X : X \to Y$ over $\overline{\tilde{y}_K}$. Thus the above $\widetilde{Y}_{\mathrm{pd}}(k)$ actually equals $\widetilde{Y}_{\mathrm{pd}} \cap \widetilde{Y}(k)$, and the relative height $\mathbf{h}^L_{X/Y}$ described in (5.2.20) coincides with the restriction of the relative height in [31, Section 4.2] to the subset $\widetilde{Y}_{\mathrm{pd}}(k)$.

Lemma 5.4. *Let A, L, $\widetilde{Y}$, $p : Y \times A \to Y$, and X be as above. Let $\widetilde{Y}_{\mathrm{pd}}(k)$ be as in* (5.2.19). *Let $\tilde{y}$ be a point in $\widetilde{Y}_{\mathrm{pd}}(k)$, and let $\tilde{y}_{\overline{K}} \in Y\left(\overline{K}\right)$ be the corresponding point via the natural inclusion $\widetilde{Y}(k) \hookrightarrow Y\left(\overline{K}\right)$. Then the following are equivalent to each other:*

(a) *Any irreducible component of $X_{\tilde{y}_{\overline{K}}} = (p|_X)^{-1}\left(\tilde{y}_{\overline{K}}\right)$ has dense small points as a closed subvariety of $p^{-1}\left(\tilde{y}_{\overline{K}}\right) \cong A$;*

(b) $\mathbf{h}^L_{X/Y}(\tilde{y}) = 0$.

Proof. Noting Remark 5.3, we immediately obtain the lemma from (5.2.20) and [31, Remark 4.5]. $\square$

5.3. **Proofs of the results.** In this subsection, we establish Theorem 5.7: Conjecture 5.1 holds for any closed subvariety of A of codimension 1. Furthermore, using this theorem, we show that Conjecture 5.1 holds for any closed subvariety of A of dimension 1 (cf. Theorem 5.12).

Before giving the proofs of the theorems, we prove a couple of technical lemmas.

Lemma 5.5. *Let A be a nowhere degenerate abelian variety over $\overline{K}$. Let $\widetilde{B}$ be an abelian variety over k, and set $B := \widetilde{B} \otimes_k \overline{K}$. Let X be a closed subvariety of $B \times A$.*

Let $\mathrm{pr}_B : B \times A \to B$ *be the first projection, and set* $Y := \mathrm{pr}_B(X)$. *Suppose that* X *has dense small points. Then the following holds.*

(1) *There exists a closed subvariety* $\widetilde{Y} \subset \widetilde{B}$ *such that* $Y = \widetilde{Y} \otimes_k \overline{K}$.
(2) *There exists a dense open subset* $\widetilde{V} \subset \widetilde{Y}$ *with the following property: For any* $\tilde{y} \in \widetilde{V}(k)$, *each irreducible component of* $\mathrm{pr}_A\left(\mathrm{pr}_B^{-1}\left(\tilde{y}_{\overline{K}}\right) \cap X\right) \subset A$ *has dense small points, where* $\tilde{y}_{\overline{K}} \in Y\left(\overline{K}\right)$ *denotes the point in* $Y\left(\overline{K}\right)$ *corresponding to* $\tilde{y}$ *via* $\widetilde{Y}(k) \hookrightarrow Y\left(\overline{K}\right)$ *and where* $\mathrm{pr}_A : B \times A \to A$ *is the second projection.*

Proof. Let L be an even ample line bundle on A. We have (1) by [31, Proposition 5.1]. Furthermore, this proposition says that there exists a dense open subset $\widetilde{V} \subset \widetilde{Y}$ such that $\widetilde{V}(k) \subset \widetilde{Y}_{\mathrm{pd}}(k)$ and such that $\mathbf{h}^L_{X/Y}(\tilde{y}) = 0$ for any $\tilde{y} \in \widetilde{V}(k)$ (note also Remark 5.3). By Lemma 5.4, it follows that for any $\tilde{y} \in \widetilde{V}(k)$, each irreducible component of $\left(\mathrm{pr}_B|_X\right)^{-1}\left(\tilde{y}_{\overline{K}}\right) \subset \{\tilde{y}_{\overline{K}}\} \times A$ has dense small points. Since pr_A induces an isomorphism $\{\tilde{y}_{\overline{K}}\} \times A \cong A$, any irreducible component of $\mathrm{pr}_A\left(\mathrm{pr}_B^{-1}\left(\tilde{y}_{\overline{K}}\right) \cap X\right) \subset A$ has dense small points. $\square$

Lemma 5.6. *Let* A *be an abelian variety over* $\overline{K}$. *Let* A_1 *be an abelian subvariety of* A, *and let* A_2 *be a quotient abelian variety of* A. *Then the following hold:*

(1) *If* A *has trivial* $\overline{K}/k$*-trace, then* A_1 *and* A_2 *have trivial* $\overline{K}/k$*-trace.*
(2) *If* A *is nowhere degenerate, then* A_1 *and* A_2 *are nowhere degenerate.*

Proof. The assertion (2) follows from [30, Lemma 7.8] immediately. To show (1), suppose that A has trivial $\overline{K}/k$-trace. Let $\widetilde{B}$ be any abelian variety over k, and set $B := \widetilde{B} \otimes_k \overline{K}$.

Let $\phi_1 : B \to A_1$ be a homomorphism. Since $A_1 \subset A$, ϕ_1 is then regarded as a homomorphism from B to A. Since A has trivial $\overline{K}/k$-trace, this homomorphism is trivial. Thus ϕ_1 is trivial, which shows that A_1 has trivial $\overline{K}/k$-trace.

To show that A_2 has trivial $\overline{K}/k$-trace, let $\phi_2 : B \to A_2$ be a homomorphism. By the Poincaré complete reducibility theorem, there exists a finite homomorphism $\psi : A_2 \to A$. Since A has trivial $\overline{K}/k$-trace, the composite $\psi \circ \phi_2 : B \to A$ is trivial. Since ψ is finite, it follows that ϕ_2 is trivial. This completes the proof. $\square$

Let A be an abelian variety $\overline{K}$. By [30, Lemma 7.9], there exists a unique abelian subvariety of A that is maximal among the nowhere degenerate abelian subvarieties. This abelian subvariety is called the *maximal nowhere degenerate abelian subvariety of* A [30, Definition 7.10].

Now, we show that Conjecture 5.1 holds for codimension 1 subvarieties.

Theorem 5.7. *Let* A *be an abelian variety over* $\overline{K}$, *and let* X *be a closed subvariety of* A *of codimension* 1. *Then if* X *has dense small points, then* X *is a special subvariety.*

Proof. We prove the theorem in three steps.

Step 1. In this step, assume that A is nowhere degenerate and has trivial $\overline{K}/k$-trace. Then we prove that X does not have dense small points. (Note that then we have the theorem for such an A, because there does not exist X that satisfies the assumption of the theorem.)

Since X is a non-trivial effective divisor on A, the linear system $|2X|$ gives a morphism $\varphi : A \to \mathbb{P}^N$. There exists a hyperplane H in $\mathbb{P}^N$ such that $\varphi^*(H) = 2X$. Let $S(X)$ be a closed subgroup of A given by $S(X)(\overline{K}) = \{a \in A(\overline{K}) \mid T_a^*(X) = X\}$, where T_a is the translate by a and "$T_a^*(X) = X$" means an equality as divisors. Let $S(X)^0$ be the connected component of $S(X)$ with $0 \in S(X)^0$, and let $\phi : A \to A_0 := A/S(X)^0$ be the quotient homomorphism. By Lemma 5.6, the abelian variety B is nowhere degenerate and has trivial $\overline{K}/k$-trace.

By [16, page 88, Remarks on effective divisors by Nori], φ factors though ϕ; i.e., there exists a finite morphism $\psi : A_0 \to \mathbb{P}^N$ such that $\varphi = \psi \circ \phi$. Set $E := \psi^*(H)$. Then E is an effective ample divisor on A_0. Furthermore, since ϕ is surjective and $2X = \varphi^*(H)$, we have $\phi(X) = E$ as closed subsets of A_0. Since A_0 is nowhere degenerate and has trivial $\overline{K}/k$-trace, Proposition 4.6 tells us that E does not have dense small points. Therefore by [29, Lemma 2.1], X does not have dense small points, either.

Step 2. We prove the theorem for the case where A is nowhere degenerate.

Suppose from here on that X has dense small points. Let $(\widetilde{A^{\overline{K}/k}}, \mathrm{Tr}_A)$ be the $\overline{K}/k$-trace of A. To ease notation, we put $\widetilde{B} := \widetilde{A^{\overline{K}/k}}$ and $B := \widetilde{B} \otimes_k \overline{K}$. Let $\mathfrak{t}$ be the image of Tr_A, and set $A_1 := A/\mathfrak{t}$. Since A is nowhere degenerate, A_1 is nowhere degenerate and has trivial $\overline{K}/k$-trace (cf. [30, Lemma 7.8 (2)] and [31, Remark 5.4]). Then by the Poincaré complete reducibility theorem, A is isogenous to $\mathfrak{t} \times A_1$. Since B is isogenous to $\mathfrak{t}$ (cf. [29, Lemma 1.4]), it follows that there exists an isogeny $\phi : A \to B \times A_1$. Set $X' := \phi(X)$, which is a closed subvariety of $B \times A_1$ of codimension 1. Since X has dense small points, so does X' (cf. [29, Lemma 2.1]). By Lemma 5.5 (1), there exists a closed subvariety $\widetilde{Y} \subset \widetilde{B}$ such that $\mathrm{pr}_B(X') = Y := \widetilde{Y} \otimes_k \overline{K}$, where $\mathrm{pr}_B : B \times A_1 \to B$ is the canonical projection. Let $\widetilde{Y}_{\mathrm{pd}}(k)$ be as on (5.2.19). By Remark 5.3 and Lemma 5.5 (2), there exists a subset $\widetilde{S} \subset \widetilde{Y}_{\mathrm{pd}}(k)$ such that $\widetilde{S}$ is dense in $\widetilde{Y}$ and such that for any $\tilde{s} \in \widetilde{S}$, each irreducible component of $\mathrm{pr}_{A_1}\left(\mathrm{pr}_B^{-1}(\tilde{s}_{\overline{K}}) \cap X'\right) \subset A_1$ has dense small points, where $\tilde{s}_{\overline{K}} \in Y\left(\overline{K}\right)$ is the point corresponding to $\tilde{s}$ via the natural map $\widetilde{Y}(k) \hookrightarrow Y\left(\overline{K}\right)$.

Note that a general fiber of $\mathrm{pr}_B|_{X'} : X' \to Y$ has dimension $\dim(A_1)$. Indeed, if this is not the case, then $\mathrm{pr}_B^{-1}(\tilde{s}_{\overline{K}}) \cap X'$ is an effective divisor on $\{\tilde{s}_{\overline{K}}\} \times A_1$ for any $\tilde{s} \in \widetilde{S}$, and hence $\mathrm{pr}_{A_1}\left(\mathrm{pr}_B^{-1}(\tilde{s}_{\overline{K}}) \cap X'\right)$ is an effective divisor on A_1 each of which irreducible components has dense small points. However, since A_1 is nowhere degenerate and has trivial $\overline{K}/k$-trace, this contradicts Step 1 above. Thus a general fiber of $\mathrm{pr}_B|_{X'} : X' \to Y$ has dimension $\dim(A_1)$.

Since $X' \subsetneq B \times A_1$, it follows that $\widetilde{Y} \neq \widetilde{B}$. Since X' has codimension 1, this means $X' = \widetilde{Y} \times_{\mathrm{Spec}(k)} A_1$. This shows that X' is a special subvariety.

Step 3. We prove the theorem for an arbitrary A.

Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. Let $A \to A' := A/\mathfrak{m}$ be the quotient homomorphism. By the Poincaré complete reducibility theorem, there exists an isogeny $\varphi : A \to A' \times \mathfrak{m}$. We set $\phi : A \to \mathfrak{m}$ to be the composite $\mathrm{pr}_{\mathfrak{m}} \circ \varphi$, where $\mathrm{pr}_{\mathfrak{m}} : A' \times \mathfrak{m} \to \mathfrak{m}$ is the canonical projection.

We claim that $\phi(X)$ is a special subvariety of $\mathfrak{m}$. If $\phi(X) = \mathfrak{m}$, then it is special trivially. Therefore, suppose that $\phi(X) \subsetneq \mathfrak{m}$. Then, since X has codimension 1 in A, so does $\phi(X)$ in $\mathfrak{m}$. Since X has dense small points, so does $\phi(X)$ (cf. [29, Lemma 2.1]). By Step 2 above, it follows that $\phi(X)$ is special.

By assumption, furthermore, X has dense small points. By [30, Theorem 7.21], it follows that X is a special subvariety of A. This completes the proof of the theorem. □

We give a couple of remarks. The following proposition is shown in Step 1 of the proof of Theorem 5.7.

Proposition 5.8. *Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace, and let X be a closed subvariety of A of codimension 1. Then X does not have dense small points.*

Remark 5.9. By Proposition 2.1, it follows from Proposition 5.8 that a non-trivial effective divisor on a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace has positive canonical height. That is a generalization of Proposition 4.6.

Remark 5.10. Let A be a nowhere degenerate abelian variety over $\overline{K}$ with trivial $\overline{K}/k$-trace. It follows from Proposition 5.8 that no non-zero effective divisor is a torsion subvariety of A, but we can show this directly, not via this proposition, in fact. To prove that by contradiction, suppose that there exists a torsion subvariety of codimension 1. Then there exists an abelian subvariety G of codimension 1. Take the quotient A/G. Then by Lemma 5.6, it is nowhere degenerate and has trivial $\overline{K}/k$-trace. On the other hand, since A/G is a nowhere degenerate elliptic curve, it follows from the well-known fact that the moduli space of elliptic curves is affine that A/G is a constant abelian variety. That is a contradiction.

Next, we consider the case where the subvariety has dimension 1. We show one more lemma.

Lemma 5.11. *Let A be an abelian variety over $\overline{K}$, and let X_1 and X_2 be closed subvarieties of A with dense small points. Then $X_1+X_2 := \{x_1+x_2 \mid x_1 \in X_1, x_2 \in X_2\}$ has dense small points.*

Proof. Note that $X_1 + X_2$ is the image of $X_1 \times X_2 \subset A \times A$ by the addition homomorphism $\alpha : A \times A \to A$. By [29, Lemma 2.4], $X_1 \times X_2$ has dense small points, and by [29, Lemma 2.1], it follows that $X_1 + X_2 = \alpha(X_1 \times X_2)$ has dense small points. □

Now, we show the dimension 1 case.

Theorem 5.12. *Let A be an abelian variety over $\overline{K}$, and let X be a closed subvariety of A of dimension 1. If X has dense small points, then it is a special subvariety.*

Proof. We prove the theorem in two steps.

Step 1. In this step, we make an additional assumption that the $\overline{K}/k$-trace of A is trivial, under which we prove that if X has dense small points, then it is a torsion subvariety.

Let $A\left(\overline{K}\right)_{tor}$ denote the set of torsion points of $A\left(\overline{K}\right)$. Consider

$$\mathfrak{S} := \left\{(\tau, A_\tau) \,\middle|\, \tau \in A\left(\overline{K}\right)_{tor},\ A_\tau \text{ is an abelian subvariety of } A \text{ with } X - \tau \subset A_\tau \right\}.$$

We take $(\tau_1, A_{\tau_1}) \in \mathfrak{S}$ such that

$$\dim\left(A_{\tau_1}\right) = \min\left\{\dim(A_\tau) \mid (\tau, A_\tau) \in \mathfrak{S}\right\}.$$

To ease notation, we set $A_1 := A_{\tau_1}$ and $X_1 := X - \tau_1$. Remark that $X_1 \subset A_1$ and that A_1 has trivial $\overline{K}/k$-trace by Lemma 5.6 (1).

We set $Y_0 := \{0\} \subset A_1$ and $Y_1 := X_1 - X_1$. For each integer $m > 1$, we define Y_m to be the sum of m copies of Y_1 in A_1, namely,

$$Y_m := \underbrace{Y_1 + \cdots + Y_1}_{m}.$$

Then, since the sequence $(Y_m)_{m \in \mathbb{Z}_{\geq 0}}$ is an ascending sequence of closed (reduced irreducible) subvarieties of A_1, this sequence is stable for large m; that is, there exists a positive integer N such that $Y_{N-1} \subsetneq Y_N = Y_{N+1} = Y_{N+2} = \cdots$. (Since $Y_0 \subsetneq Y_1$, we remark $N \geq 1$.) Then we note that Y_N is an abelian subvariety of A_1. Indeed, $Y_N \neq \emptyset$, and we see $Y_N - Y_N \subset Y_{2N} = Y_N$.

Now, suppose that X has dense small points. We would like to show that $X_1 = A_1$. Note that $X_1 = X - \tau_1$ also has dense small points.

Claim 5.12.21. We have $Y_N = A_1$.

Proof. Let $\psi : A_1 \to A_1/Y_N$ be the quotient homomorphism. Since A_1 has trivial $\overline{K}/k$-trace, so does A_1/Y_N by Lemma 5.6 (1). Fix a point $x_1 \in X_1\left(\overline{K}\right)$. Then $X_1 - x_1 \subset Y_1 \subset Y_N$. It follows that $\psi(X_1 - x_1) = \{0\}$, and hence $\psi(X_1) = \{\psi(x_1)\}$. Since X_1 has dense small points, so does $\{\psi(x_1)\}$ by [29, Lemma 2.1], which means that $\psi(x_1)$ is a torsion point. Since a surjective morphism of abelian varieties over an algebraically closed field induces a surjective morphism between the torsion points, there exists a $\tau' \in A_1\left(\overline{K}\right)_{tor}$ such that $\psi(\tau') = \psi(x_1)$. Then $\psi(X_1 - \tau') = \{0\}$, which means $Y_N \supset X_1 - \tau' = X - (\tau_1 + \tau')$. We note that $\tau_1 + \tau'$ is a torsion point. By the minimality of $\dim(A_1)$ among $\{\dim(A_\tau) \mid (\tau, A_\tau) \in \mathfrak{S}\}$, we have $\dim(Y_N) \geq \dim(A_1)$. Since $Y_N \subset A_1$, this proves $Y_N = A_1$. ☐

Claim 5.12.22. We have $N = 1$.

Proof. We argue by contradiction. Suppose that $N \geq 2$. Then $Y_1 \subset Y_{N-1} \subsetneq A_1$. Since X_1 has dense small points, so does Y_{N-1} by Lemma 5.11. Note that Y_{N-1} is not an abelian subvariety. Indeed, if this is not the case, then $Y_{N-1} + Y_{N-1} \subset Y_{N-1}$, so that $Y_N = Y_{N-1} + Y_1 \subset Y_{N-1} + Y_{N-1} \subset Y_{N-1}$, which contradicts the choice of N.

Since $A_1 = Y_N = Y_{N-1} + Y_1$ (cf. Claim 5.12.21) and $1 \leq \dim Y_1 \leq 2$, we have $1 \leq \operatorname{codim}(Y_{N-1}, A_1) \leq 2$. In fact, we see that $\operatorname{codim}(Y_{N-1}, A_1) = 2$. To show that by contradiction, suppose that $\operatorname{codim}(Y_{N-1}, A_1) = 1$. Since Y_{N-1} has dense small points, Y_{N-1} is then a special subvariety by Theorem 5.7. Since the $\overline{K}/k$-trace of A_1 is trivial by the assumption in this step and since $0 \in Y_{N-1}$, it follows that Y_{N-1} is an abelian subvariety. However, this contradicts what we have noted above. Thus $\operatorname{codim}(Y_{N-1}, A_1) = 2$.

Set $Z := Y_{N-1} + X_1$. Then by Lemma 5.11, it is a closed subvariety of A_1 with dense small points. Furthermore, since $\operatorname{codim}(Y_{N-1}, A_1) = 2$ and $Z - X_1 = Y_N = A_1$, we have $\operatorname{codim}(Z, A_1) = 1$. Again by Theorem 5.7 together with the fact that the $\overline{K}/k$-trace of A_1 is trivial, it turns out that Z is a torsion subvariety. Thus there exist an abelian subvariety G of A_1 and a torsion point $\tau' \in A_1\left(\overline{K}\right)$ such that $Z = G + \tau'$. Since $0 \in Y_{N-1}$, we have $X_1 \subset Y_{N-1} + X_1 = Z$. It follows that $X - (\tau' + \tau_1) = X_1 - \tau' \subset G$. By the minimality of $\dim(A_1)$, this implies that $\dim(G) \geq \dim(A_1)$. Since $G \subset A_1$, we obtain $G = A_1$, and thus $Z = A_1$. However, this contradicts $\operatorname{codim}(Z, A_1) = 1$. Thus we obtain $N = 1$. ☐

By Claim 5.12.21 and Claim 5.12.22, we have $X_1 - X_1 = A_1$. Since $\dim(X_1) = 1$, we have $1 \leq \dim(A_1) \leq 2$.

Claim 5.12.23. We have $\dim(A_1) = 1$.

Proof. We argue by contradiction. Suppose that $\dim(A_1) \neq 1$, or equivalently, $\dim(A_1) = 2$. Then X_1 is a divisor on A_1. Since X_1 has dense small points, it follows from Theorem 5.7 that X_1 is a torsion subvariety, where we note that the $\overline{K}/k$-trace of A_1 is trivial. Since $\dim(X_1) = 1$, this means that the translate of X_1 by some torsion point is an abelian subvariety of dimension 1, which contradicts the definition of A_1. □

Since $X_1 \subset A_1$, we have $X_1 = A_1$ by Claim 5.12.23. Thus $X = X_1 + \tau_1 = A_1 + \tau_1$, which is a torsion subvariety of A. This completes Step 1.

Step 2. We prove the general case.

Let $\mathfrak{t}$ be the image of the $\overline{K}/k$-trace homomorphism (cf. Section 2.1). Let $\phi : A \to A/\mathfrak{t}$ be the quotient homomorphism, and set $X' := \phi(X)$. Since X has dense small points , so does X' by [29, Lemma 2.1]. Since $A/\mathfrak{t}$ has trivial $\overline{K}/k$-trace (cf. [31, Remark 5.4]), it follows from Remark 5.2 and Step 1 that there exist an abelian subvariety $G \subset A/\mathfrak{t}$ and a torsion point $\tau' \in A/\mathfrak{t}$ such that $X' = G + \tau'$. Note $\dim(G) \leq 1$. Since ϕ induces a surjective homomorphism between the torsion points, we take a torsion point $\tau \in A\left(\overline{K}\right)$ with $\phi(\tau) = \tau'$.

Set $Y := X - \tau$ and $B := \phi^{-1}(G)$. Then $Y \subset B$, and since $\phi : A \to A/\mathfrak{t}$ is smooth and has connected fibers, B is an abelian subvariety of A with $\mathfrak{t} \subset B$. Let $\mathfrak{n}$ be the maximal nowhere degenerate abelian subvariety of B, and let $\mathfrak{s}$ be the image of the $\overline{K}/k$-trace homomorphism of B. By the universality of the $\overline{K}/k$-trace of B, we have $\mathfrak{t} \subset \mathfrak{s}$. (The other inclusion holds by the universality of the $\overline{K}/k$-trace of A also, and thus they are equal in fact.) Since then $\dim(\mathfrak{n}/\mathfrak{s}) \leq \dim(\mathfrak{n}/\mathfrak{t}) \leq \dim(G) \leq 1$, the geometric Bogomolov conjecture holds for $\mathfrak{n}/\mathfrak{s}$ (cf. Remark 5.2). By [31, Theorem 5.5], it follows that the conjecture holds for B. Since Y has dense small points, Y is therefore a special subvariety. Thus $X = Y + \tau$ is also a special subvariety, which completes the proof of the theorem. □

5.4. **Bogomolov conjecture for curves.** In this subsection, let C be a smooth projective curve of genus $g \geq 2$ over $\overline{K}$. Let J_C be the Jacobian variety of C. Fix a divisor D of degree 1 on C, and let $\jmath_D : C \hookrightarrow J_C$ be the embedding defined by $\jmath_D(x) := x - D$. For any $P \in J_C\left(\overline{K}\right)$ and any $\epsilon \geq 0$, set

$$B_C(P, \epsilon) := \left\{ x \in C\left(\overline{K}\right) \,\middle|\, ||\jmath_D(x) - P||_{NT} \leq \epsilon \right\},$$

where $||\cdot||_{NT}$ is the semi-norm arising from the Néron-Tate height on J_C associated to a symmetric theta divisor.

Recall that C is said to be *isotrivial* if there exists a projective curve $\widetilde{C}$ over k with $C \cong \widetilde{C} \otimes_k \overline{K}$. As is remarked in [30, Section 8], we obtain the following theorem as a consequence of Theorem 5.12.

Theorem 5.13 (Theorem 1.2, Bogomolov conjecture for curves over any function field)**.** *Assume that C is non-isotrivial. Then, for any $P \in J_C\left(\overline{K}\right)$, there exists an $r > 0$ such that $B_C(P, r)$ is finite.*

Thus we conclude that the Bogomolov conjecture for curves over any function field holds.

Remark 5.14. In the above theorem, we consider the Néron-Tate height associated to a symmetric theta divisor, but this theorem also holds with respect to the canonical height associated to any even ample line bundle because of [29, Lemma 2.1].

6. Consequences of the main results

6.1. **Geometric Bogomolov conjecture for abelian varieties.** In this subsection, we make a contribution to the geometric Bogomolov conjecture (Conjecture 1.5), which claims that a closed subvariety X of A has dense small points if and only if X is a special subvariety. Since we know the "if" part holds (cf. [29]), the "only if" part is the problem.

We begin with some background of the geometric Bogomolov conjecture. The version of the conjecture over number fields was first established by Zhang in [33], which claims that for a closed subvariety X of an abelian variety over a number field, X has dense small points if and only if X is a torsion subvariety. Zhang's theorem generalizes Ullmo's theorem, which is the case of a curve in its Jacobian, and its proof is inspired by the proof of Ullmo. Further, Moriwaki generalized the result to the case where K is a finitely generated field over $\mathbb{Q}$, with respect to certain arithmetic heights. Remark that Moriwaki's arithmetic heights are not the classical heights over function fields.

Over function fields with respect to the classical heights, no analogous results were known for a while even after Zhang's theorem, but in 2007, Gubler proved the following result. Here A is *totally degenerate* at $v \in M_{\overline{K}}$ if A has torus reduction at v.

Theorem 6.1 (Theorem 1.1 of [6]). Let K be a function field. Let A be an abelian variety over $\overline{K}$. Assume that A is totally degenerate at some $v \in M_{\overline{K}}$. Let X be a closed subvariety of A. Then X has dense small points if and only if X is a torsion subvariety.

The geometric Bogomolov conjecture is inspired by Gubler's theorem. Remark that his theorem is equivalent to the geometric Bogomolov conjecture for an abelian variety which is totally degenerate at some place because then a special subvariety is a torsion subvariety.

Although the geometric Bogomolov conjecture for abelian varieties is still an open problem, there are partial solutions. Recently, in [30] and [31], we obtain results which generalize Gubler's theorem. Here we would like to recall them. Let A be an abelian variety over $\overline{K}$, and let $\mathfrak{m}$ be the maximal nowhere degenerate abelian subvariety of A. We remark that if A is totally degenerate at some place, then $\mathfrak{m} = 0$ (cf. [30, Introduction]), but the converse does not hold in general. Let $\mathfrak{t}$ be the image of the trace homomorphism $\mathrm{Tr}_A : \widetilde{A}^{\overline{K}/k} \otimes_k \overline{K} \to A$. Since $\widetilde{A}^{\overline{K}/k} \otimes_k \overline{K}$ is nowhere degenerate, $\mathfrak{t} \subset \mathfrak{m}$ (cf. [30, Lemma 7.8 (2)]), and $\mathfrak{m}/\mathfrak{t}$ is a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace (cf. [30, Lemma 7.8 (2)] and [31, Remark 5.4]). Following [30], we obtain in [31] the following result.

Theorem 6.2 (cf. Theorem 1.5 of [31]). Let A be an abelian variety over $\overline{K}$. Then the following are equivalent to each other:

(a) The geometric Bogomolov conjecture holds for A;
(b) The geometric Bogomolov conjecture holds for $\mathfrak{m}/\mathfrak{t}$.

Theorem 6.2 shows that the geometric Bogomolov conjecture for abelian varieties is equivalent to that for nowhere degenerate abelian varieties with trivial $\overline{K}/k$-trace. In particular, Theorem 6.2 proves that the conjecture holds for A with $\dim(\mathfrak{m}/\mathfrak{t}) = 0$ (cf. [30, Theorem 1.4]). Since $\mathfrak{m} = \mathfrak{t} = 0$ for a totally degenerate abelian variety A, this generalizes Gubler's theorem.

Our main results of this paper give a further generalization of the above results. First, as a consequence of Theorems 5.7 and 5.12, we obtain the following result (see also Remark 5.2).

Corollary 6.3. *Let A be an abelian variety over $\overline{K}$ with $\dim(A) \leq 3$. Then the geometric Bogomolov conjecture holds for A.*

Then, by Theorem 6.2, we generalize Corollary 6.3 as follows.

Corollary 6.4. *Let A be an abelian variety over $\overline{K}$. Let $\mathfrak{m}$ be the maximal nowhere degenerate abelian variety, and let $\mathfrak{t}$ be the image of the trace homomorphism. Assume that $\dim(\mathfrak{m}) \leq \dim(\mathfrak{t}) + 3$. Then the geometric Bogomolov conjecture holds for A.*

Remark 6.5. Since the trace homomorphism is finite, we have $\dim(\mathfrak{t}) = \dim(\widetilde{A}^{\overline{K}/k})$.

The geometric Bogomolov conjecture remains open even now for a higher dimensional abelian variety, but we can see that the conjecture holds for some kind of them. For example, let A be an abelian variety of dimension 4 over $\overline{K}$. Then we have proved that the geometric Bogomolov conjecture holds for A unless it is a nowhere degenerate abelian variety with trivial $\overline{K}/k$-trace.

6.2. **Manin-Mumford conjecture and the Bogomolov conjecture.** In this subsection, we remark that our main results give an alternative proof of the Manin-Mumford conjecture in a positive characteristic in a special setting, by mentioning the relationship between this conjecture and the geometric Bogomolov conjecture.

Let F be an algebraically closed field. Originally, the Manin-Mumford conjecture claims that under the assumption of $\mathrm{char}(F) = 0$, a smooth projective curve of genus at least 2 over F embedded in its Jacobian should have only finitely many torsion points of the Jacobian.

This was first proved by Raynaud in [20] in the following generalized form. Here, we say that a closed subvariety X of an abelian variety A has *dense torsion points* if $X \cap A(F)_{tor}$ is dense in X.

Theorem 6.6. Assume that $\mathrm{char}(F) = 0$. Let A be an abelian variety over F, and let X be a closed subvariety of A. Assume that $\dim(X) = 1$. If X has dense torsion points, then X is a torsion subvariety.

Further, Raynaud proved in [21] that the above statements hold for any dimensional X. We should remark that Moriwaki proved in [15] that Raynaud's theorem is a consequence of his theorem on the arithmetic Bogomolov conjecture over a field finitely generated over $\mathbb{Q}$.

Now, we consider what happens when $\mathrm{char}(F) = p > 0$. In this case, there exists an obvious counterexample: If F is an algebraic closure of a finite field and X is any closed subvariety of A, then any F-point of X is a torsion point, and thus X always has dense torsion points. However, up to such an influence of subvarieties which can be defined over finite fields, one may expect that a similar statement should also hold in positive characteristics. We have the following precise result.

Theorem 6.7. Let F be an algebraically closed field with $\mathrm{char}(F) > 0$, and let k be the algebraic closure in F of the prime field of F. Let A be an abelian variety over F, and let X be a closed subvariety of A. If X has dense torsion points, then there exist an abelian subvariety G of A, a closed subvariety $\widetilde{Y}$ of $\widetilde{A}^{F/k}$, and a torsion point $\tau \in A(F)$ such that $X = G + \mathrm{Tr}_A^{F/k}(\widetilde{Y} \otimes_k F) + \tau$, where $(\widetilde{A}^{F/k}, \mathrm{Tr}_A^{F/k})$ is the F/k-trace of A.

This theorem is due to the following authors: In 2001, Scanlon gave a sketch of the model-theoretic proof of this theorem [23]. In 2004, Pink and Roessler gave an algebro-geometric proof [18]. In 2005, Scanlon gave a detailed model-theoretic proof in [24] based on the argument in [23]. Note that in those papers, they prove a generalized version for semiabelian varieties A, in fact.

Here, we mention that Theorem 6.7 can be deduced from the geometric Bogomolov conjecture for A, as Moriwaki did in the case of characteristic 0. Let F, k, A, and X be as in Theorem 6.7. Then there exist $t_1, \ldots, t_n \in F$ such that A and X can be defined over $K := k(t_1, \ldots, t_n)$; that is, there exist an abelian variety A_0 over K and a closed subvariety X_0 of A_0 such that $A = A_0 \otimes_K F$ and $X = X_0 \otimes_K F$. Further, there exists a normal projective variety $\mathfrak{B}$ over k with function field K. Let $\overline{K}$ be the algebraic closure of K in F, and set $A_{\overline{K}} := A_0 \otimes_K \overline{K}$ and $X_{\overline{K}} := X_0 \otimes_K \overline{K}$. Let $\mathcal{H}$ be an ample line bundle on $\mathfrak{B}$. Then we have a notion of height over K, and we can consider the canonical height on $A_{\overline{K}}$ associated to an even ample line bundle. Suppose that X has dense torsion points. Then $X_{\overline{K}}$ has dense torsion points and hence has dense small points. Then if we assume the geometric Bogomolov conjecture, it follows that $X_{\overline{K}}$ should be a special subvariety, and this should show the conclusion of Theorem 6.7.

Since the geometric Bogomolov conjecture is still open, the above argument does not give a new proof of Theorem 6.7. However, we can actually deduce the theorem in the following special cases:

(1) $\dim(X) = 1$ or $\mathrm{codim}(X, A) = 1$ (from Theorems 5.12, 5.7);
(2) $\dim(A) \leq 3$ (from Corollary 6.3).

In particular, we have recovered the positive characteristic version of Theorem 6.6.

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Institute for Liberal Arts and Sciences, Kyoto University, Kyoto, 606-8501, Japan
E-mail address: `yamaki.kazuhiko.6r@kyoto-u.ac.jp`