JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 1, Pages 119–169
S 0894-0347(05)00502-3
Article electronically published on September 13, 2005

LONG ARITHMETIC PROGRESSIONS IN SUMSETS: THRESHOLDS AND BOUNDS

E. SZEMERÉDI AND V. VU

1. Overview

One of the main tasks of additive number theory is to examine structural properties of sumsets. For a set A of integers, the sumset $lA = A + \cdots + A$ consists of those numbers which can be represented as a sum of l elements of A:

$$lA = \{a_1 + \cdots + a_l | a_i \in A_i\}.$$

Closely related and equally interesting notion is that of l^*A, which is the collection of numbers which can be represented as a sum of l different elements of A:

$$l^*A = \{a_1 + \cdots + a_l | a_i \in A_i, a_i \neq a_j\}.$$

Among the most well-known results in all of mathematics are Vinogradov's theorem, which says that $3\mathbb{P}$ ($\mathbb{P}$ is the set of primes) contains all sufficiently large odd numbers, and Waring's conjecture (proved by Hilbert, Hardy and Littlewood, Hua, and many others), which asserts that for any given r, there is a number l such that $l^*\mathbb{N}^r$ ($\mathbb{N}^r$ denotes the set of rth powers) contains all sufficiently large positive integers (see [29] for an excellent exposition concerning these results).

In recent years, a considerable amount of attention has been paid to the study of finite sumsets. Given a finite set A and a positive integer l, the natural analogue of Vinogadov-Waring results is to show that under proper conditions, the sumset lA (l^*A) contains a long arithmetic progression.

Let us assume that A is a subset of the interval $[n] = \{1, \ldots, n\}$, where n is a large positive integer. The concrete problem we would like to address is to estimate the minimum length of the longest arithmetic progression in lA (l^*A) as a function of l, n, and $|A|$. We denote this function by $f(|A|, l, n)$ ($f^*(|A|, l, n)$), following the notation in [13]. Many estimates for $f(|A|, l, n)$ have been discovered by Bourgain, Freiman, Halberstam, Green, Ruzsa, and Sárközy (see Section 3), but most of these results focus on sets with very high density, namely $|A|$ is close to n. Estimating $f^*(|A|, l, n)$ seems much harder, and not much was known prior to our study.

Received by the editors October 6, 2004.

2000 *Mathematics Subject Classification.* Primary 11B25, 11P70, 11B75.

Key words and phrases. Sumsets, arithmetic progressions, generalized arithmetic progressions, complete and subcomplete sequences, inverse theorems.

The first author is supported by an NSF grant.

The second author is an A. Sloan Fellow and is supported by an NSF Career Grant.

In this paper, we solve both problems almost completely for a wide range of l and $|A|$. Our study reveals a surprising fact that the functions $f(|A|, l, n)$ and $f^*(|A|, l, n)$ are not continuous and admits a threshold rule. We have successfully located the threshold points within constant errors and established the asymptotic behavior of the functions between consecutive threshold points. It has also turned out, during our study, that the sum l^*A is indeed fundamentally harder to attack than its counterpart lA.

Central to our study is the development of a new, purely combinatorial, method. This method is totally different from harmonic analysis methods used by most researchers and seems quite flexible. For instance, it is easy to extend our results in many directions. Moreover, the method carries us far beyond our original aim of estimating lengths of arithmetic progressions, leading to more general theorems about proper generalized arithmetic progressions (GAPs).

Our results also have some interesting applications. In particular, we settle two forty-year-old conjectures of Erdös [8] and Folkman [14] (respectively) concerning infinite arithmetic progressions.

Let us now present a brief introduction to the content of our paper:

- In Section 2 we present the notion of GAPs and state Freiman's famous inverse theorem, both of which play a crucial role in our study. In Section 3, we first describe some earlier results on the topic. Next, we present a construction which suggests a conjecture about the length of the longest arithmetic progression in lA. It would be important to keep this construction in mind as it motivates lots of our arguments later on. The first main result of Section 3 confirms the conjecture motivated by the construction. This result, among others, reveals the surprising fact that $f(|A|, l, n)$ is not continuous and admits a threshold behavior. There are many threshold points, and we are able to locate them within a constant factor. The second main result, which refines the first one, provides a more general and complete picture. We can prove that lA not only contains long arithmetic progressions, but it also contains large proper generalized arithmetic progressions (a regular arithmetic progression is a special proper generalized arithmetic progression of rank one). In Section 4 we prove these two results. The first four subsections of Section 4 are devoted to the development of a variety of tools, through which we could establish a connection between our study and inverse theorems of Freiman type. Exploiting this connection, we complete the proofs in the final two subsections. This concludes the first part of the paper.
- The second part of the paper consists of Section 5 and Section 6. In Section 5, we generalize the results in Section 3 to sums of different sets. Instead of considering lA, we consider the sum $A_1 + \cdots + A_l$, where $|A|_1 = \cdots = |A_l| = |A|$. Thanks to the flexibility of our method, we can extend the results of Section 3 to this setting in a relatively simple manner. Also in this part of the paper we discuss an application which settles a conjecture posed by Folkman in 1966. This conjecture was considered by Erdös and Graham ([9], Section 6) to be the most important problem in the study of subcomplete sequences. An infinite sequence is subcomplete if its partial sums contain an infinite arithmetic progression. Folkman conjectured

that a sufficiently dense sequence of positive integers (with possible repetitions) is subcomplete. In Section 6, we first work out a sufficient condition for subcompleteness and next use the results in Section 5 to show that a sufficiently dense sequence should satisfy this condition.

- Sections 7, 8, and 9 form the third part of the paper. This part contains our strongest result, the proof of which is also the most technical. The heart of this part of the paper is Theorem 7.1, which extends the results in Section 3 to the sumset l^*A. The proof comprises several phases. In the first phase, we prove a structural property of a set A where l^*A does not contain a generalized arithmetic progression as large as we desire. This property, which might be of independent interest, shows that such a set A contains a very rigid subset which almost looks like a generalized arithmetic progression. The verification of the structural lemma occupies most of Section 7. Section 8 contains the rest of the proof, whose core consists of an observation about proper GAPs (subsection 8.3) and a variant of the so-called tiling technique, introduced in an earlier paper [28]. Section 9 discusses a conjecture of Erdös (posed in 1962) which is related to the above-mentioned conjecture of Folkman. This conjecture was proved in an earlier paper [28] using a special case of the main result in Section 7, but here we give a shorter proof using the general condition worked out in Section 6. Several other applications of the main result of this part will appear in future papers [30, 31].
- The last part of the paper contains Section 10, in which we extend all previous results to finite fields. We assume that n is a prime and consider arithmetic progressions modulo n. This modification will lead to a natural change in the statement of the results, but the proofs remain basically the same. We conclude this part by mentioning an application concerning the problem of counting zero-sum-free sets.

The paper contains several new technical ingredients, some of which (such as the study of proper GAPs in Sections 3 and 7 and the rank reduction argument used in Sections 3 and 4) would be of independent interest. Our writing benefits from two earlier papers [27, 28], which established several partial results and launched the foundation of our study. Many ideas from these two papers will be used here, frequently in more general and more comprehensible forms.

2. Inverse theorems

A generalized arithmetic progression (GAP) of rank d is subset Q of $\mathbb{Z}$ of the form $\{a + \sum_{i=1}^d x_i a_i | 0 \leq x_i \leq n_i\}$; the product $\prod_{i=1}^d n_i$ is its volume, and we denote it by $\mathrm{Vol}(Q)$. In fact, as two different GAPs might represent the same set, we always consider GAPs together with their structures. The set $(a_1, \dots, a_d)$ is called the *difference* set of Q.

Freiman's famous inverse theorem [12] asserts that if $|A + A| \leq c|A|$, where c is a constant, then A is a dense subset of a generalized arithmetic progression of constant rank. In fact, the statement still holds in a slightly more general situation, when one considers $A + B$ instead of $A + A$. This was shown by Ruzsa [24], who gave a very elegant proof which was different from Freiman's.

Theorem 2.1. *For every positive constant c there is a positive integer d and a positive constant k such that the following holds. If A and B are two subsets of $\mathbb{Z}$*

with the same cardinality and $|A+B| \leq c|A|$, then A is a subset of a generalized arithmetic progression P of rank d with volume at most $k|A|$.

The most recent estimate on k (as a function of c) is due to Chang [5]. In our paper, however, we shall be more concerned with the best value of d (see Lemma 4.9). The following result is a simple consequence of Fremain's theorem and Plüneke's theorem (for the statement of Plüneke's theorem see, e.g., [24]).

Theorem 2.2. *For every positive constant c there is a positive integer d and a positive constant k such that the following holds. If A and B are two subsets of $\mathbb{Z}$ with the same cardinality and $|A+B| \leq c|A|$, then $A+B$ is a subset of a generalized arithmetic progression P of rank d with volume at most $k|A|$.*

For the special case when c is relatively small, one can set $d = 1$. The following is a consequence of another theorem of Freiman [12].

Lemma 2.3. *The following holds for all sufficiently large m. If A is a set of integers of cardinality m and $|A+A| \leq 2.1m$, then A is a subset of an arithmetic progression of length $1.1m$.*

Again, we can replace $A+A$ by $A+B$. The following is a corollary of a result by Lev and Smeliansky (Theorem 6 of [19]).

Lemma 2.4. *The following holds for all sufficiently large m. If A and B are two sets of integers of cardinality m and $|A+B| \leq 2.1m$, then A is a subset of an arithmetic progression of length $1.1m$.*

Both Lemmas 2.3 and 2.4 are relatively simple and do not require the inverse theorem to be proved.

3. Long arithmetic progressions in lA

3.1. **Some previous results.** Problems concerning arithmetic progressions in sumsets are non-trivial, and not too many results are known. In the following, we describe some of the main results in this area. Bourgain [3] proved that if $|A| = \delta n$ where δ is a positive constant, then $2A$ contains an arithmetic progression of length $e^{\epsilon \log^{1/3} n}$, where ϵ is a positive constant depending on δ. Freiman, Halberstam, and Ruzsa [10] consider sumsets modulo a prime and proved

Theorem 3.2. *Let n be a prime and A a set of residues modulo n, $|A| = \gamma n$, $0 < \gamma < 1$, may depend on n. Let l be a positive integer at least 3. Then lA contains an arithmetic progression (modulo n) of length $\Omega(\gamma n^{\frac{1}{16}\gamma^{l/(l-2)}})$.*

Notice that Theorem 3.2 is stated for any γ, but it is really efficient only when γ is relatively large. Indeed, if one wants to have $\gamma n^{\frac{1}{16}\gamma^{l/(l-2)}} \geq 1$, one needs to set

$$\gamma = \Omega(\frac{1}{\ln n}).$$

So Theorem 3.2 does not give a non-trivial bound in the case $|A| = o(\frac{n}{\ln n})$. Bourgain's result and Theorem 3.2 have recently been improved by Green [16], but the applicable range does not change.

Prior to our study, the only result (that we know of) which applies to sets with relatively small cardinality is the following theorem, proved by Sárközy [25].

Theorem 3.3. *There are positive constants c and C such that the following holds. If A is a subset of $[n]$ and l is a positive integer such that $l|A| \geq Cn$, then lA contains an arithmetic progression of length $cl|A|$.*

Answering a question of Sárközy, Lev [20] has shown that one can set C equal to 2, which is the optimal value.

It is clear that Theorem 3.3 is sharp, up to a constant factor. Let A be the set of all positive integers from 1 to $|A|$. Then lA is the set of all positive integers from l to $l|A|$.

The main result of this section gives a sharp estimate for a wide range of $|A|$ and l, including Theorem 3.3 as a special case. More importantly, our proof reveals the structures of those sets A whose sumsets lA do not contain a very long arithmetic progression. In the next subsection, we describe the construction that motivates our result.

To conclude this subsection, let us mention that the proofs of all results mentioned in this paper, with the exception of Sárközy's proof, are analytic, making heavy use of harmonic analysis, and are very different from the proofs in this paper.

3.4. **Sudden jumps.** Our first crucial observation is that the statement of Theorem 3.3 no longer holds when $l|A|$ becomes a little bit less than n. The following construction shows that there is a set $A \subset [n]$ and a number l such that $l|A| \approx n/4$ while the length of the longest arithmetic progression in lA is only $O(l|A|^{1/2})$ (here and later $\approx$ means "approximately").

The construction. Let $A = \{p_1x_1 + p_2x_2 | 1 \leq x_1 \leq m\}$, where $p_1 \approx p_2 \approx \frac{n}{2m}$ are two primes and $p_2 > m$. It is convenient to think of A as a square in the two dimensional lattice $\mathbb{Z}^2$. A point (x_1, x_2) corresponds to the number $p_1x_1 + p_2x_2$. It is easy to show that this correspondence is one-to-one. Indeed

$$p_1x_1 + p_2x_2 = p_1x_1' + p_2x_2'$$

implies that

$$p_1(x_1 - x_1') = p_2(x_2' - x_2),$$

which is impossible because of divisibility and the fact that $|x_1 - x_1'| < m < p_2$. Thus, $|A| = m^2$. Let $l = \frac{n}{(4+\epsilon)|A|} = \frac{n}{(4+\epsilon)m^2}$, where ϵ is an arbitrary positive constant. We have

$$lA = \{p_1x_1 + p_2x_2 | l \leq x_1 \leq lm\}.$$

Let P be an arithmetic progression (AP) in lA, we are going to show that the coordinates of the elements of P also form an AP of the same length. Thus $|P|$ is at most the length of an edge of $l|A|$, which is less than $lm = l|A|^{1/2}$. Observe that

$$p_2 \approx n/2m \geq 2lm \text{ since } l = n/(4+\epsilon)|A| = n/(4+\epsilon)m^2.$$

Consider three consecutive terms in P, $z + z^{''} = 2z'$. Write $z = p_1x_1 + p_2x_2$. We have

$$(p_1x_1 + p_2x_2) + (p_1x_1^{''} + p_2x_2^{''}) = 2(p_1x_1' + p_2x_2'),$$

which implies

$$p_1(x_1 + x_1^{''} - 2x_1') = -p_2(x_2 + x_2^{''} - 2x_2'),$$

which is again impossible as

$$|x_1 + x_1^{''} - 2x_1'| < 2lm \leq p_2.$$

Next, we generalize the above construction to higher dimensions.

The general construction. Let d be a constant positive integer at least 2, and let δ be a small positive constant. Consider two numbers $|A|$ and l satisfying $l^{d-1}|A| \leq \frac{1-\delta}{2d}n$. We shall construct a set A of cardinality $|A|$ such that the longest arithmetic progression in lA has length $l|A|^{1/d}$.

Set $a = \lfloor \frac{(1-\delta/3)n}{d|A|^{1/d}} \rfloor$ and $b = \lfloor (\frac{n}{d|A|^{1/d}})^{1/(d-1)} \rfloor$. Set $b_1 = 0, b_2 = 1$, and if $d \geq 3$, then set $b_i = \lfloor b^{(i-2)/(d-1)} \rfloor$ for all $3 \leq i \leq d$. Finally set $a_i = a + b_i$. It is routine to verify that for a sufficiently large n

$$(1 - \delta/3)a^{1/(d-1)} \geq 2l|A|^{1/d}. \tag{1}$$

Consider the set

$$A = \{\sum_{i=1}^{d} a_i x_i | 1 \leq x_i \leq |A|^{1/d}\}$$

(for convenience we assume that $|A|^{1/d}$ is an integer). The term $\frac{1-\delta}{d}$ in the definition of a guarantees that P is a subset of $[n]$. It is convenient to view both A and lA as d-dimensional integral boxes. The edges of lA form arithmetic progressions of length $l|A|^{1/d}$. Similarly to the case $d = 2$, we are going to prove the following two claims.

Claim 3.5. *lA does not contain an arithmetic progression of length greater than $l|A|^{1/d}$.*

Claim 3.6. The cardinality of A is $|A|$.

Proof of Claim 3.5. Consider an arithmetic progression P in lA, and let $z, z^{'}, z^{''}$ be three consecutive elements of P'. We have $z + z^{''} = 2z^{'}$. Write $z = \sum_{i=1}^d a_i x_i, z' = \sum_{i=1}^d a_i x_i^{'}$ and $z^{''} = \sum_{i=1}^d a_i x_i^{''}$. It follows that $\sum_{i=1}^d (x_i + x_i^{''} - 2x_i^{'})a_i = 0$. Notice that $1 \leq x_i, x_i^{'}, x_i^{''} \leq l|A|^{1/d}$, so $|x_i + x_i^{''} - 2x_i^{'}| < 2l|A|^{1/d}$, for all i's.

Next, we show that the diophantine equation $\sum_{i=1}^d r_i a_i = 0$ cannot have non-trivial roots with small absolute values; namely, $|r_i| < 2l|A|^{1/d}$ cannot hold simultaneously for all i's. Consider a non-trivial root $\{r_1, \ldots, r_d\}$. There are two cases.

(I) $\sum_{i=1}^d r_i = 0$. By the definition of the a_i's, it follows that $\sum_{i=1}^d r_i b_i = 0$ and d should be at least 3. Let j be the largest index where $r_j \neq 0$. It is easy to see that $j \geq 3$. On the other hand, by the definition of the b_i's, for any $j \geq 3$

$$\max_{1 \leq i \leq d} |r_i| \geq \frac{b_j}{\sum_{i=1}^{j-1} b_i} \geq a^{1/(d-1)} \geq 2l|A|^{1/d}, \tag{2}$$

where the last inequality is from (1).

(II) $\sum_{i=1}^d r_i \neq 0$. In this case, it is obvious that

$$\max_{1 \leq i \leq d} |r_i| \geq \frac{a}{\sum_{i=1}^d b_i} \geq (1 - \delta)a^{1/(d-1)} \geq 2l|A|^{1/d}. \tag{3}$$

By the previous facts, we can conclude that $x_i + x_i^{''} - 2x_i^{'} = 0$ for all i's. So for each i, the coordinates of z_i form an arithmetic progression. This implies that the length of P could be at most the length of the "edges" of A, which is $l|A|^{1/d}$. □

From the previous proof, it is obvious that if $\sum_{i=1}^{d} a_i x_i = \sum_{i=1}^{d} a_i x_i'$ for $1 \leq x_i, x_i' \leq |A|^{1/d}$ for all $1 \leq i \leq d$, then $x_i = x_i'$ for all i's. This implies that the cardinality of A is $|A|$, proving Claim 3.6.

This construction plays a very important role in the whole paper. It not only leads us to the statements of our theorems, but also motivates many of our arguments.

The sudden jumps. For the sake of simplicity, let us consider l and n fixed and view $f(|A|, l, n)$ as a function of $|A|$ (we call this function $g(|A|)$). The special case $d = 2$ shows that if $|A| \leq \frac{1-\delta}{4}\frac{n}{l}$, then $g(|A|)$ is upper bounded by $l|A|^{1/2}$. This and Theorem 3.3 imply that $g(|A|)$ admits a dramatical change in order of magnitude somewhere near the point $\frac{n}{l}$. If $|A| \geq C\frac{n}{l}$ for some sufficiently large constant C, then $g(|A|)$ (up to a multiplicative constant) behaves like $l|A|$. On the other hand, if $|A| \leq \frac{1-\delta}{4}\frac{n}{l}$, then $g(|A|)$ is upper bounded by $l|A|^{1/2}$. This indicates that $g(|A|)$ is not a continuous function, and its behavior must follow a threshold rule.

The general construction suggests that n/l is not the only threshold (a place where $g(|A|)$ jumps). Assume, for a moment, that we could prove that close to the left of n/l, $g(|A|)$ behaves like $l|A|^{1/2}$. This behavior, however, cannot continue to hold with $|A|$ getting significantly smaller than n/l. Indeed, once $|A|$ becomes less than $\frac{1-\delta}{6}\frac{n}{l^2}$, then $g(|A|)$ is upper bounded by $l|A|^{1/3}$. Thus, another threshold should occur around the point $\frac{n}{l^2}$. Motivated by this reasoning, one would conjecture that there is a threshold around $\frac{n}{l^d}$ for any fixed positive integer d. To the right of the threshold, $g(|A|)$ behaves like $l|A|^{1/d}$, while to the left it behaves like $l|A|^{1/(d+1)}$.

3.7. $g(|A|)$ **must jump.** Our first main result confirms the above conjecture.

Theorem 3.8. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. For any positive integers n and l and any set $A \subset [n]$ satisfying $l^d|A| \geq Cn$, lA contains an arithmetic progression of length $cl|A|^{1/d}$.*

Corollary 3.9. *For any fixed positive integer d there are positive constants C_1, C_2, c_1, and c_2 depending on d and ϵ such that whenever $\frac{C_1 n}{l^d} \leq |A| \leq \frac{C_2 n}{l^{d-1}}$,*

$$c_1 l|A|^{1/d} \leq f(|A|, l, n) \leq c_2 l|A|^{1/d}.$$

Let us again consider $f(|A|, l, n)$ as a function $g(|A|)$ of $|A|$, assuming n and l are fixed. It is more convenient to view $g(|A|)$ on a logarithmic scale. For this purpose, let us define $x = \ln|A|$ and $y(x) = \ln g(|A|)$. Corollary 3.9 implies

Corollary 3.10. *For any fixed positive integer d there are constants C_1, C_2, c_1, and c_2 depending on d such that whenever $\ln n - d\ln l + C_1 \leq x \leq \ln n - (d-1)\ln l + C_2$,*

$$\frac{1}{d}x + \ln l + c_1 \leq y(x) \leq \frac{1}{d}x + \ln l + c_2.$$

The values of the constants C_1, C_2, c_1, c_2 in this corollary are, of course, different from the values of C_1, C_2, c_1, c_2 in Theorem 3.8. Corollary 3.10 determines the value of $y(x)$ up to a constant additive term for all x except for a few intervals of constant lengths. An exceptional interval is a neighborhood of a threshold point $\ln n - d\ln l = \ln\frac{n}{l^d}$ and is of the form $[\ln n - d\ln l + C_2(d-1), \ln n - d\ln l + C_1(d)]$, which has length $C_1(d) - C_2(d-1)$. Here we write $C_1(d)$ and $C_2(d-1)$ instead of C_1 and C_2 to emphasize the dependence on d and $d-1$, respectively.

The above results locate the thresholds within constant factors. It would be nice to find the exact locations of these thresholds.

Question. *What are the exact values of the constants C and c in Theorem* 3.8*?*

The case $d = 1$ was treated by Lev in [20]. For general d, our construction shows that $C(d)$ is at least $(1 - o(1))/2d$.

3.11. **A stronger theorem about generalized arithmetic progressions.** Theorem 3.8 is only the tip of the iceberg, and we are going to extend it in various directions. In the first extension, we show that Theorem 3.8 is a consequence of a stronger theorem about GAPs.

In order to guess what we may say about the possible existence of GAPs in lA, let us go back to the construction. Observe that the constructed sumset lA contains not only an arithmetic progression of length $l|A|^{1/d}$, but also a proper GAP of rank d and cardinality $\Omega(l^d|A|)$. The arithmetic progression of length $l|A|^{1/d}$ we talked about is actually an edge of this GAP. Thus, our first guess is, naturally, that lA contains a GAP of rank d and cardinality $\Omega(l^d|A|)$. This guess is, nevertheless, false. To see this, notice that if we let A in the construction be a GAP of dimension $d' < d$ with appropriate parameters, then lA is a GAP of dimension d' of cardinality $\Omega(l^{d'}|A|)$ which is much less than $\Omega(l^d|A|)$ (it is interesting to note that in this case lA contains an arithmetic progression of length $\Omega(l|A|^{1/d'}) \gg \Omega(l|A|^{1/d})$). So, the strongest statement one could make is that lA contains a proper GAP of rank d' and cardinality $\Omega(l^{d'}|A|)$ for some integer $1 \le d' \le d$. This turns out to be the truth.

Theorem 3.12. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. For any positive integers n and l and any set $A \subset [n]$ satisfying $l^d|A| \ge Cn$, lA contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer* $1 \le d' \le d$.

The other main results of this paper, Theorems 5.1, 7.1, 8.13, and 10.3, are extensions of this theorem in various directions.

To conclude this subsection, let us point out that both Theorem 3.8 and Theorem 3.12 are invariant under affine transformations. Instead of assuming that A is a subset of $[n]$, we can assume that A is a subset of an arithmetic progression of length n. In fact, for technical reasons, we will frequently assume that A contains 0.

3.13. **More about generalized arithmetic progressions.** Consider a GAP $Q = \{a + \sum_{i=1}^d x_i a_i | 0 \le x_i \le n_i\}$. It is convenient to consider Q together with the box $B_Q = \{(x_1, \dots, x_d) | \ 0 \le x_i \le n_i\}$ of d-dimensional vectors and the following map Φ from $\mathbb{Z}^d$ to $\mathbb{Z}$:

$$\Phi(x_1, \dots, x_d) = a + \sum_{i=1}^d x_i a_i.$$

The volume of Q is the geometrical volume of the d-dimensional box spanned by B_Q:

$$\mathrm{Vol}(Q) = \mathrm{Vol}(B_Q) = \prod_{i=1}^d n_i.$$

We say that Q is *proper* if $\Phi(B_Q)$ is injective. In this case the cardinality of Q is $\prod_{i=1}^{d}(n_i+1) = |B_Q|$. It is trivial that

$$|Q| \leq 2^d \mathrm{Vol}(Q), \tag{4}$$

and if Q is proper, then

$$\mathrm{Vol}(B_Q) < |B_Q| \leq 2^d \mathrm{Vol}(B_Q). \tag{5}$$

If Q is not proper, then there are two vectors u and w in B_Q such that $\Phi(u) = \Phi(w)$. The vector $v = u - w$ is called a *vanishing* vector. By linearity, it is clear that if v is vanishing, then $\Phi(v) = 0$ and $\Phi(v+u) = \Phi(u)$ for any $u \in \mathbb{Z}^d$.

In the following we specify some rules used in calculations involving GAPs.

Addition. We only add two GAPs with the same difference set, and the result is a GAP with this difference set. For instance, if $P = \{a + a_1x_1 + \cdots + a_dx_d | 0 \leq x_i \leq m_i\}$ and $Q = \{b + a_1x_1 + \cdots + a_dx_d | 0 \leq x_i \leq n_i\}$, then

$$P + Q = \{(a+b) + a_1x_1 + \cdots + a_dx_d | 0 \leq x_i \leq m_i + n_i\}.$$

Substraction is defined similarly.

Multiplication. For a GAP P, we have $2P = P + P$ and $lP = (l-1)P + P$.

Division. Consider a GAP $P = \{a + a_1x_1 + \cdots + a_dx_d | 0 \leq x_i \leq m_i\}$. We say P is *normal* if $a = 0$. In this case, we define

$$\frac{1}{s}P = \{a_1x_1 + \cdots + a_dx_d | 0 \leq x_i \leq m_i/s\}.$$

All of our arguments concerning GAPs are invariant with respect to affine transformation (shiftings in particular), so we could (and shall) automatically assume that a GAP is normal when it is involved in division.

3.14. **Some simple tricks.** In this subsection, we describe several simple tricks which we use frequently throughout the paper.

As C can be set arbitrarily large, we can sacrifice constant factors in many arguments. So we are going to make several assumptions, whose "prices" are only constant factors, which are very convenient for the proofs.

Divisibility. By increasing the value of C, we can assume that l is a power of two. Indeed, if we replace l by the closest power of two, then the magnitude of l decreases by at most 2. Similarly, once we have a GAP of constant rank and all we care about is the volume of this GAP, up to a constant factor, then we can assume that the lengths of the edges are divisible by 2 (or by any fixed integer). This latter assumption is convenient for divisions. For instance, whenever we need to divide a GAP P by a constant s, we shall always assume that the lengths of the edges of P are divisible by s.

Passing to subsets. In many situations, it is useful to assume that a certain set, say X, has a certain property. On the other hand, we can only prove that X has a subset X' with the desired property. However, when X' has constant density in X, we can frequently assume that X has the desired property, again by increasing the value of C.

A graph with small degrees contains a large independent set. A graph consists of a set V of vertices and a set E of edges, where an edge is a pair of two different vertices. The degree of a vertex v is the number of edges containing v. If (u, v) is an edge, then u is a neighbor of v and vice versa. A subset of V is called *independent* if it does not contain any edge. We are going to use the following simple fact from graph theory.

Fact 3.15. Let G be a graph on n vertices. Assume that any vertex of G has degree at most d. Then G contains an independent set of size $n/(d+1)$.

Proof. Let I be a maximal independent set. Since I is maximal, the neighbors of the vertices in I and I together cover the vertex set of G. Since the vertices of I have at most $d|I|$ neighbors, it follows that

$$d|I| + |I| \geq n,$$

proving the claim. □

The above fact implies that if G does not contain an independent set of size s, then G has a vertex with degree at least n/s.

4. Proofs of Theorem 3.8 and Theorem 3.12

This section has six subsections. In the first four subsections we develop a variety of tools. The proofs of Theorem 3.8 and Theorem 3.12 are presented in the last two subsections.

Let us start with a sketch of the proof of Theorem 3.8. Consider the sequence

$$A, 2A, 4A, ..., lA$$

(without loss of generality we can assume that l is a power of 2). Since lA is a subset of the interval $[ln]$, $|lA|$ is at most ln. This implies that the ratio $|2^{i+1}A|/|2^iA|$ cannot always be large. In particular, there is a constant K such that $|2^{i+1}A| \leq K|2^iA|$ holds for some index i less than $\log_2 l$. On the other hand, $2^{i+1}A = 2^iA + 2^iA$, so by applying Freiman's theorem, we can deduce that 2^iA is a dense subset of a GAP P with constant rank.

Let us assume, for a moment, that 2^iA has density one in P, namely, $2^iA = P$. Thus 2^iA contains a long arithmetic progression B of length at least $(\mathrm{Vol}P)^{1/\mathrm{rank}(P)}$. As i is less than $\log_2 l$, lA contains an even longer arithmetic progression of length at least $\frac{l}{2^i}|B|$.

In order to carry out this scheme, we first need to show that assuming $2^iA = P$ is not oversimplifying. This will be carried out in the second subsection, where we show that at the cost of constant factors we can think of a dense subset of a GAP as the whole set.

With the aid of this assertion, it is now not so hard to prove that lA contains an arithmetic progression of length $l|A|^\epsilon$ for some small ϵ. In order to optimize ϵ, we need to optimize K and the rank of P. The optimal value of K is easy to guess while the optimal value of the rank of P will be provided by a result of Bilu [2], which is a part of his proof of Freiman's theorem.

Now comes the last, and perhaps most intriguing, point. Even with these optimal parameters, we could not obtain the bound claimed in the theorem (however, we can obtain a weaker theorem proved in an earlier paper [27]). To fill in the gap, we need to prove certain properties of non-proper and proper GAPs. These properties lead us to Lemma 4.13, which is the main lemma of the proof. The verification of this lemma requires the preparation carried out throughout the first three subsections.

Now let us say something about the proof of Theorem 3.12. The first step is to realize that we can assume that 2^iA is not only a GAP, but also a proper one.

The sumset lA contains a multiple of this GAP. The trouble is that a multiple of a proper GAP does not need to be proper. What saves us here is a technique called "rank reduction". The heart of this technique is an argument which shows that under certain circumstances a multiple of a proper GAP either is proper or contains a proper GAP of strictly smaller rank and comparable cardinality. Thus if we fail to complete our task in the first attempt, we can pass to a proper GAP with smaller rank and try again. The GAP we start with has a constant rank so sooner or later we must be done. The reader should notice that this approach, in spirit, is consistent with the statement of Theorem 3.12, which confirms the existence of a GAP of rank d' where d' is an undetermined quantity between 1 and d. This value d' is exactly where the rank reduction terminates.

4.1. **A property of non-proper GAPs.** Let us consider the ratio between the cardinality and the volume of a GAP P. Assume that P has the form $P = \{a + a_1x_1 + \cdots + a_dx_d | 0 \leq x_i \leq n_i\}$, where all $n_i's \geq 1$. The volume of P is $\prod_{i=1}^d n_i$. If P is proper, then its cardinality is $\prod_{i=1}^d (n_i + 1)$ and the ratio in question is $\prod_{i=1}^d (1 + \frac{1}{n_i})$, which is a number between 1 and 2^d. For a non-proper GAP, it is safe to say that the ratio is less than 2^d, but it could still be larger than 1. We are going to show, nevertheless, that if P is a sufficiently large multiple of a non-proper GAP, then this ratio is bounded from above by any fixed positive constant ϵ.

Lemma 4.2. *For any positive constants ϵ and d there is a constant g such that the following holds. If a GAP Q of rank d is not proper, then $|gQ| \leq \epsilon \mathrm{Vol}(gQ)$. Moreover,*

$$|2Q| \leq (1 - \frac{1}{2^{d+1}})|2B_Q|.$$

In the proof, we are going to use terminologies introduced in subsection 3.13. The reader may want to read this subsection again before checking the proof.

Proof of Lemma 4.2. We can assume that $Q = \{x_1a_1 + \cdots + x_da_d | 0 \leq x_i \leq n_i\}$. We consider Q together with the box B_Q and the canonical map Φ from B_Q to Q. Since Q is not proper, there is a vanishing vector v where $-n_i \leq v_i \leq n_i$ for all $i = 1, \dots, d$. Without loss of generality, we can assume that the first d' coordinates of v are positive and the remaining ones are non-positive. Thus $0 < v_i \leq n_i$ for $i = 1, \dots, d'$ and $-n_i \leq v_i \leq 0$ for $d' < i \leq d$.

Let $h < g$ be sufficiently large integers, and let B' be the set of vectors w in gB_Q such that $w + v, w + 2v, \dots, w + hv$ are also in gB_Q. As v is vanishing, $\Phi(w) = \Phi(w + v) = \cdots = \Phi(w + hv)$. It follows that

$$|gQ| \leq |gB_Q \backslash B'| + \frac{1}{h+1}|B'| = |gB_Q| - \frac{h}{h+1}|B'|, \tag{6}$$

which implies

$$|gQ| \leq (1 - \frac{h}{h+1}\frac{|B'|}{|gB_Q|})|gB_Q| \leq 2^d(1 - \frac{h}{h+1}\frac{|B'|}{|gB_Q|})\mathrm{Vol}(gB_Q), \tag{7}$$

where in the last inequality we use the trivial fact that $|gB_Q| \leq 2^d \mathrm{Vol}(gB_Q)$ (see (4)). Next we bound $|B'|$ from below. A vector w is surely in B' if $0 \leq w_i \leq (g-h)n_i$ for $i \leq d'$ and $hn_i \leq w_i \leq gn_i$ for $d' < i \leq d$. Thus the cardinality of B' is at least

$\prod_{i=1}^{d}\big((g-h)n_i+1\big)$. Moreover, $|gB_Q| \le \prod_{i=1}^{d}(gn_i+1)$, so

$$\frac{h}{h+1}\frac{|B'|}{|gB_Q|} \ge \frac{h}{h+1}\prod_{i=1}^{n}\frac{\big((g-h)n_i+1\big)}{gn_i+1}. \tag{8}$$

For any given ϵ, d we could choose g and h (depending only ϵ and d) so that

$$\frac{h}{h+1}\prod_{i=1}^{d}\frac{\big((g-h)n_i+1\big)}{gn_i+1} \ge 1-\epsilon/2^d$$

holds for any positive integers n_i's. With this choice of g and h, the rightmost formula in (7) is thus at most $\epsilon\mathrm{Vol}(gQ)$, proving the first statement of the lemma. To verify the second statement, set $g=2$ and $h=1$. We obtain

$$|2Q| \le (1-\frac{1}{2}\frac{|B'|}{|gB_Q|})|gB_Q| \le (1-\frac{1}{2}\prod_{i=1}^{d}\frac{n_i+1}{2n_i+1})|2B_Q|. \tag{9}$$

The product $\prod_{i=1}^{d}\frac{n_i+1}{2n_i+1}$ is larger than $\frac{1}{2^d}$ so it follows that

$$|2Q| \le (1-\frac{1}{2^{d+1}})|2B_Q|, \tag{10}$$

completing the proof. □

4.3. **The proper filling lemma.** In this subsection, we present several lemmas which allow us to think of a dense subset of a GAP as the whole set, at the cost of constant factors. The first such lemma was proved in [27].

Lemma 4.4. *For any positive constant γ and any positive integer d there is a constant positive integer h and a positive constant γ' depending on γ and d such that the following holds. If P is a generalized arithmetic progression of rank d and B is a subset of P such that $|B| \ge \gamma\mathrm{Vol}(P)$, then hB contains a generalized arithmetic progression of rank d with cardinality at least $\gamma'|B|$.*

We call this lemma the "filling lemma", as our motivation is to fill out a complete GAP. Next, we strengthen this lemma by adding the requirement that the GAP contained in hB must be proper.

Lemma 4.5. *For any positive constant γ and any positive integer d there is a constant positive integer h and a positive constant γ' depending on γ and d such that the following holds. If P is a generalized arithmetic progression of rank d and B is a subset of P such that $|B| \ge \gamma\mathrm{Vol}(P)$, then hB contains a proper generalized arithmetic progression of rank d with cardinality at least $\gamma'|B|$.*

We shall, naturally, refer to Lemma 4.5 as the "proper filling lemma". The proof of Lemma 4.5 combines Lemma 4.4 with the result of the previous subsection.

Proof of Lemma 4.5. By Lemma 4.4, hB contains a GAP Q with cardinality $\Omega(|B|)$. It suffices to show that Q contains a proper GAP of the same rank with cardinality $\Omega(|Q|)$. As $h=O(1)$, $\mathrm{Vol}(hP)=O(\mathrm{Vol}(P))=O(|B|)$, so we can assume that

$$|Q| \ge \gamma_1\mathrm{Vol}(hP) \tag{11}$$

for some positive constant γ_1.

Let g be a large constant integer. Without loss of generality we can assume that $Q=\{x_1a_1+\cdots+x_da_d|0\le x_i\le n_i\}$ and n_i is divisible by g. Let ϵ be a positive constant smaller than γ_1 and consider the GAP $Q' = \frac{1}{g}Q$. If Q' is proper, then we are done as

$$|Q'| \geq \mathrm{Vol}(Q') = \Omega(\mathrm{Vol}(Q)) = \Omega(|Q|).$$

We next show that Q' is indeed proper given that g is sufficiently large. Assume otherwise. Choosing g as in Lemma 4.2, we have

$$|Q| = |gQ'| \leq \epsilon \mathrm{Vol}(gQ') = \epsilon \mathrm{Vol}(Q) \leq \epsilon \mathrm{Vol}(hP) < \gamma_1 \mathrm{Vol}(hP), \tag{12}$$

which contradicts (11). This completes the proof. □

4.6. (δ, d)**-sets.** We begin this subsection with an important definition.

Definition 4.7. A set A is a (δ, d)-set if one can find a GAP Q of rank d such that $B = Q \cap A$ satisfies $|B| \geq \delta \max\{|A|, \mathrm{Vol}(Q)\}$.

The filling lemmas tell us that a (δ, d)-set (where both δ and d are constant) can be treated as a GAP of rank d, if we are allowed to sacrifice constant factors.

Lemma 4.8. *For any positive constants δ and d there are positive constants g and γ such that the following holds. If A is a (δ, d)-set, then gA contains a proper GAP of rank d with cardinality at least $\gamma|A|$.*

Now we are going to present another lemma, which supplies a sufficient condition for a set to be a (δ, d)-set. In order to motivate this lemma, let us go back to Freiman's inverse theorem. Freiman's theorem shows that if $|A+A| \leq c|A|$, then A is a dense subset of a GAP P of rank $d = d(c)$. As we mentioned at the beginning of this section, the optimal value of d is critical to us. Observe that if A is a proper GAP of dimension d, then $|A + A| \leq 2^d|A|$. So, one may wonder whether one can set $d = \lfloor \log_2 c \rfloor$. Unfortunately, Freiman's theorem is not true with this value of d (the best-known bound is $d = \lfloor c \rfloor$). On the other hand, if we can afford to sacrifice constant factors, then we can actually obtain this optimal value of d. To be more precise, if $|A+A| \leq c|A|$, then a constant fraction of A is contained in a GAP P of ranked $d = \lfloor \log_2 c \rfloor$ with small volume. The following lemma is a consequence of Theorem 1.3 of [2].

Lemma 4.9. *For any positive constants ϵ and d there is a positive constant δ such that the following holds. If $|A + A| \leq (2^d - \epsilon)|A|$, then A is a (δ, d)-set.*

This lemma is a coproduct of the proof of Freiman's theorem given by Bilu in [2].

4.10. **Rank reduction.** Now we are in a position to develop the so-called rank reduction technique, mentioned in the beginning of this section. This technique plays an important role not only in the proofs of Theorems 3.8 and 3.12, but also in the proof of Theorem 7.1.

The rank reduction technique allows us to pass from one GAP to another which has strictly smaller rank and comparable cardinality. We are going to present several lemmas which constitute the technique. The first lemmas is as follows.

Lemma 4.11. *For any positive constant d there is a positive constant δ such that the following holds. If a GAP Q of rank d is proper but $2Q$ is not, then $2Q$ is a $(\delta, d-1)$-set.*

Proof of Lemma 4.11. Applying the second statement of Lemma 4.2 to $2Q$, we have that

$$|4Q| = |2(2Q)| \le (1 - \frac{1}{2^{d+1}})|4B_Q| \le (1 - \frac{1}{2^{d+1}})4^d|B_Q|, \tag{13}$$

where in the last inequality we used the fact that $|4B_Q| \le 4^d|B_Q|$. Since Q is proper, $|B_Q| = |Q|$. It follows that

$$|4Q| < (1 - \frac{1}{2^{d+1}})4^d|Q| \le (2^d - \gamma)^2|Q|, \tag{14}$$

for some constant $\gamma = \gamma(d)$. It follows that either $|2Q| \le (2^d - \gamma)|Q|$ or $|4Q| \le (2^d - \gamma)|2Q|$. In the first case Q is a $(\delta, d-1)$-set; in the second case $2Q$ is a $(\delta, d-1)$-set (both statements follow immediately from Lemma 4.9). But Q is a translation of a subset of $2Q$, so in both cases $2Q$ is a $(\delta, d-1)$-set (notice that the three δ's in the last two sentences might have different values). □

The previous lemma and Lemma 4.8 together yield

Lemma 4.12. *For any positive constant d there are positive constants g and γ such that the following holds. If a GAP Q of rank d is proper but $2Q$ is not, then gQ contains a proper GAP of rank $(d-1)$ with cardinality at least $\gamma|Q|$.*

We are now ready to present the main lemma of the proofs of Theorems 3.8 and 3.12.

Lemma 4.13. *For any positive constants ϵ and d there are positive constants c and γ such that the following holds. Let Q be a proper GAP of rank d, and assume that there are positive integers $l_1 = 2^{s_1}$ and m satisfying $l_1Q \subset [m]$ and $l_1^d|Q| \ge cm$. Then there is a positive integer $l_1' = 2^{s_1'} < l_1$ such that $l_1'Q$ contains a proper GAP Q' of rank $(d-1)$ where $|Q'| \ge \gamma l_1'^d|Q|$.*

Proof of Lemma 4.13. Consider the sets $Q_0 = Q$, $Q_i = 2Q_{i-1}$, for $i = 1, \dots, s_1 - h_1 = s_2$, where h_1 is the largest integer satisfying $2^{dh_1+d} < c$. If Q_i were proper for all i, then $|Q_i| > \mathrm{Vol}(Q_i)$ and $\mathrm{Vol}(Q_i) = 2^d\mathrm{Vol}(Q_{i-1})$, and this would imply that

$$|Q_{s_2}| > \mathrm{Vol}(Q_{s_2}) = 2^{ds_2}\mathrm{Vol}(Q_0) \ge \frac{l_1^d}{2^{dk_1}}\frac{|Q|}{2^d} \ge \frac{l^d|Q|}{2^{dk_1+d}} \ge \frac{cm}{2^{dk_1+d}} > m, \tag{15}$$

which is impossible as we assume $l_1Q \subset [m]$. (In the second inequality we used the fact that $\mathrm{Vol}(Q_0) = \mathrm{Vol}(Q) \ge \frac{|Q|}{2^d}$.) Therefore, there is some i between 1 and s_2 for which Q_i is not proper. Let j be the smallest such i. Thus, Q_{j-1} is proper and $Q_j = 2Q_{j-1}$ is not. By Lemma 4.12, there are constants h_2 and γ_1 such that h_2Q_{j-1} contains a proper GAP Q' of rank $(d-1)$ with cardinality at least $\gamma_1|Q_{j-1}|$. Without loss of generality we can assume that h_2 is a power of 2, $h_2 = 2^{h_3}$. By increasing c, we can assume that $h_1 > h_3$, which guarantees that $l_1' = h_22^j \le l_1$. The set $l_1'Q = h_2Q_{j-1}$ contains a proper GAP Q' of rank $(d-1)$ and cardinality

$$|Q'| \ge \gamma_1|Q_{j-1}| = \gamma_12^{(j-1)d}|Q| \ge \frac{\gamma_1}{h_2^d}l_1'^d|Q| = \gamma l_1'^d|Q|, \tag{16}$$

where $\gamma = \frac{\gamma_1}{h_2^d}$, concluding the proof. □

4.14. **Proof of Theorem 3.8.** Before starting the proof, let us mention that all constants (γ_1, γ_2, etc.) in the proof depend on d, but they do not depend on C. By setting C sufficiently large, we can satisfy all relations required between these constants. Without loss of generality, we can assume l is a power of two, $l = 2^s$, where s is sufficiently large. Consider the set sequence $A_0 = A$, $A_{i+1} = 2A_i$. We first need the following fact, which asserts that for some i significantly smaller than $s = \log_2 l$, the ratio $|A_i|/|A_{i-1}|$ is not too large.

Fact 4.15. There is some $i \le \frac{d+1}{d+3/2}s$ such that $|A_i| \le 2^{d+3/2}|A_{i-1}|$.

Proof of Fact 4.15. Assume otherwise. Then

$$|A_{\frac{d+1}{d+3/2}s}| \ge 2^{(d+3/2)\frac{d+1}{d+3/2}s}|A_0| = 2^{(d+1)s}|A| = l^{d+1}|A| \ge Cln, \tag{17}$$

a contradiction as $A_{\frac{d+1}{d+3/2}s}$ is a subset of $[ln]$ (C is set to be larger than 1). The proof of the claim is completed. □

Let s_1 be the first index where $|A_{s_1+1}| \le 2^{d+3/2}|A_{s_1}|$. Lemmas 4.8, 4.9, and 4.5 imply that there are constants g_1 and γ_1 depending only on d such that $2^{g_1}A_{s_1}$ contains a proper GAP Q of rank $d+1$ and cardinality at least $\gamma_1|A_{s_1}|$. By the definition of s_1

$$|A_{s_1}| \ge 2^{(d+3/2)s_1}|A|,$$

so

$$|Q| \ge \gamma_1 2^{(d+3/2)s_1}|A|.$$

By setting C sufficiently large, we can assume that s is sufficiently large so that $s \ge s_1 + g_1$ (notice that $s_1 \le \frac{d+1}{d+3/2}s$). This implies that $\frac{l}{2^{s_1+g_1}}Q$ is a subset of lA. Next we apply Lemma 4.13 to Q with $m = ln$, $l_1 = \frac{l}{2^{s_1+g_1}}$, and $d+1$ instead of d. In order to verify the conditions of this lemma, observe that

$$l_1^{d+1}|Q| \ge \frac{l^{d+1}}{2^{(s_1+g_1)(d+1)}}\gamma_1 2^{(d+3/2)s_1}|A| \ge \gamma_1 lcn 2^{s_1/2 - g_1(d+1)}. \tag{18}$$

Again by assuming that C is large, we could guarantee that the condition of Lemma 4.13 is met. Lemma 4.13 implies that we have a proper GAP $Q' \subset l_1'Q = 2^{s_1'}Q$ of rank d with cardinality at least

$$\gamma_2 2^{s_1'(d+1)}|Q| \ge \gamma_2\gamma_1 2^{(d+1)(s_1+s_1')}2^{s_1/2}|A|, \tag{19}$$

where $s_1 + s_1' \le s$. The GAP $P = 2^{s-s_1-s_1'}Q'$ is a subset of $2^sA = lA$, and its volume is

$$\begin{aligned} 2^{d(s-s_1-s_1')}\mathrm{Vol}(Q') \ge 2^{d(s-s_1-s_1')}\frac{|Q'|}{2^d} &\ge 2^{d(s-s_1-s_1'-1)}\gamma_1\gamma_2 2^{(d+1)(s_1+s_1')}2^{s_1/2}|A| \\ &= \gamma_1\gamma_2 2^{ds}|A|2^{3s_1/2+s_1'-d} \\ &\ge \frac{\gamma_1\gamma_2}{2^d}l^d|A|. \end{aligned}$$

Since P has rank d, its longest edge forms an AP of length at least

$$\Big(\frac{\gamma_1\gamma_2}{2^d}l^d|A|\Big)^{1/d} = \Omega(l|A|^{1/d}),$$

completing the proof of Theorem 3.8. □

Remark. The reader may notice that in this proof we used the estimate on the cardinality of Q', but we did not use the fact that Q' is proper. The properness of Q', however, is critical in the next proof.

4.16. **Proof of Theorem 3.12.** Without loss of generality, we can assume that $0 \in A$. Consider Q' as in the proof of Theorem 3.8. Again by increasing C, we may assume that $s - s_1 - s_1'$ is lower bounded by a sufficiently large constant. Consider the GAP $Q^{''} = 2^{s-s_1-s_1'-g_2}Q'$, where g_2 is a large constant satisfying $s - s_1 - s_1' - g_2 \geq 0$. Since $0 \in A$, $Q^{''}$ is a subset of lA. Moreover, as Q' and $Q^{''}$ are of ranked d, we have, using inequality (19), that

$$\begin{aligned}\mathrm{Vol}(Q^{''}) &\geq 2^{(s-s_1-s_1'-g_2)d}\gamma_2\gamma_1 2^{(d+1)(s_1+s_1')}2^{s_1/2}|A| \\ &= \gamma_1\gamma_2 2^{sd}2^{\frac{3s_1}{2}+s_1'-g_2d}|A| \\ &= \Omega(l^d|A|).\end{aligned}$$

We are going to examine two cases:

Case 1: $Q^{''}$ is proper. In this case lA contains the proper GAP $Q^{''}$ of rank d and volume $\Omega(l^d|A|)$. So we are done by setting $d' = d$.

Case 2: $Q^{''}$ is not proper. Now we make crucial use of the fact that Q' is proper. The properness of Q' implies that there is a positive integer $s_2 \leq s - s_1 - s_1' - g_2 \leq s$ such that $\frac{1}{2^{s_2}}Q^{''}$ is proper. As usual, we choose s_2 to be the smallest such integer, which implies that $\frac{1}{2^{s_2-1}}Q^{''} = \frac{2}{2^{s_2}}Q^{''}$ is not proper. Applying Lemma 4.12 to $\frac{1}{2^{s_2}}Q^{''}$ we obtain a GAP $Q^{'''}$ of rank $d-1$ and volume

$$\Omega(\mathrm{Vol}(\frac{1}{2^{s_2}}Q^{''})) = \Omega(\frac{1}{2^{ds_2}}\mathrm{Vol}(Q^{''}) = \Omega(\frac{1}{2^{ds_2}}l^d|A|).$$

Furthermore, there is a constant g_3 such that $Q^{''''} = 2^{s_2-g_3}Q^{'''}$ is a subset of lA. The GAP $Q^{''''}$ has rank $d-1$ and volume

$$2^{(s_2-g_3)(d-1)}\mathrm{Vol}(Q^{'''}) = \Omega(2^{s_2(d-1)}\frac{1}{2^{ds_2}}l^d|A|) = \Omega(2^{-s_2}l^d|A|).$$

Since $s \geq s_2$, $2^{-s_2} \geq 2^{-s} = l^{-1}$. Thus, the volume of $Q^{''''}$ is $\Omega(l^{d-1}|A|)$. Now if $Q^{''''}$ is proper, then we are done by setting $d' = d-1$. Otherwise we repeat the analysis of Case 2 to obtain a GAP of rank $d-2$ and so on. This repetition cannot continue forever, so sooner or later we must obtain a proper GAP of some rank $d' < d$ which satisfies the claim of the theorem. □

5. Sums of different sets

The goal of this section is to generalize the results in Section 3 by considering the sum of different sets, instead of the sum of the same sets. Given l sets $A_1, \dots, A_l$, we define

$$A_1 + \cdots + A_l = \{a_1 + \cdots + a_l | a_i \in A_i, 1 \leq i \leq l\}.$$

We obtain the following generalization of Theorem 3.12.

Theorem 5.1. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let $A_1, \dots, A_l$ be subsets of $[n]$ of size $|A|$ where l and $|A|$ satisfy $l^d|A| \geq Cn$. Then $A_1 + \cdots + A_l$ contains a GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer $1 \leq d' \leq d$.*

The following corollary generalizes Theorem 3.8.

Corollary 5.2. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let $A_1, \dots, A_l$ be subsets of $[n]$ of size $|A|$ where l and $|A|$ satisfy $l^d|A| \geq Cn$. Then $A_1 + \cdots + A_l$ contains an arithmetic progression of length $cl|A|^{1/d}$.*

Theorem 5.1 has a nice application. In Section 6, we use this theorem to confirm a conjecture of Folkman posed in 1966.

5.3. **The basic idea.** The basic idea behind the proof of Theorem 5.1 is the following. Given the sets $A_1, \dots, A_l$ as in Theorem 5.1, we are going to show that there are numbers l', n' and a set A' such that

- $l'A'$ is a subset of $A_1 + \cdots + A_l$; A' is a subset of $[n']$.
- $l', n', |A'|$ satisfy the conditions of Theorem 3.12. (This can be done by setting the constant C in Theorem 5.1 much larger than the constant C in Theorem 3.12.)
- $(l')^{d'}|A'| = \Omega(l^{d'}|A|)$ for all $1 \le d' \le d$.

An application of Theorem 3.12 to the triple (l', n', A') immediately implies the statement of Theorem 5.1.

The proof of Theorem 5.1 uses a technical lemma, Lemma 5.8. This lemma provides a sufficient condition for the existence of a sumset of form $l'A'$ in a sumset of different sets. The verification of this lemma requires extensions of the filling lemmas described in Section 4. These extensions are the topic of the next subsection.

5.4. **Filling with different sets.** In the proof of Theorem 5.1, we shall need the following lemma, which generalizes Lemma 4.4 the way Theorem 5.1 generalizes Theorem 3.8. This lemma was proved in an earlier paper [28].

Lemma 5.5. *For any positive constant γ and positive integer d, there is a positive constant γ' and a positive integer g such that the following holds. If $X_1, \dots, X_g$ are subsets of a generalized arithmetic progression P of rank d and $|X_i| \ge \gamma \mathrm{Vol}(P)$, then $X_1 + \cdots + X_g$ contains a generalized arithmetic progression Q of rank d and cardinality at least $\gamma' \mathrm{Vol}(P)$. Moreover, the distances of Q are multiples of the distances of P.*

One can further strengthen this lemma by requiring Q to be proper. The proof is similar to the proof of the proper filling lemma, Lemma 4.5.

Lemma 5.6. *For any positive constant γ and positive integer d, there is a positive constant γ' and a positive integer g such that the following holds. If $X_1, \dots, X_g$ are subsets of a generalized arithmetic progression P of rank d and $|X_i| \ge \gamma \mathrm{Vol}(P)$, then $X_1 + \cdots + X_g$ contains a proper generalized arithmetic progression Q of rank d and cardinality at least $\gamma' \mathrm{Vol}(P)$. Moreover, the distances of Q are multiples of the distances of P.*

Later on, we shall refer to Lemmas 5.5 and 5.6 as the general filling and general proper filling lemmas, respectively.

5.7. **The main lemma of Theorem 5.1.** We are now in a position to present and prove the main lemma of the proof of Theorem 5.1.

Lemma 5.8. *For every positive constant c there are positive constants ϵ and d depending on c such that the following holds. If the sets $X_1, \dots, X_l$, each of cardinality $|X|$, satisfy $|X_1 + X_i| \le c|X|$ for all $2 \le i \le l$, then there is a proper GAP Q of rank at most d and cardinality at least $\epsilon|X|$ and a number $l' \ge \epsilon l$ such that the sum $X_1 + \cdots + X_l$ contains a translation of $l'Q$.*

Proof of Lemma 5.8. The condition $|X_1+X_i| \le c|X|$ and Freiman's theorem imply that X_1 is contained in a GAP R with constant rank and volume $O(|X|)$. Consider X_i, for some $2 \le i \le l$. We say that two elements x and y of X_i are equivalent if $x-y \in R-R$. It is trivial that if x and y are not equivalent, then $x+X_1$ and $y+X_1$ are disjoint sets. Since $|X_1+X_i| \le c|X|$ where $|X| = |X_1| = |X_i|$, the number of equivalent classes is at most c. It follows that there is a class with cardinality $\Omega(|X|)$; let us call this class Y_i. The class Y_i is a translation of a subset Z_i (of constant density) of R. The hidden constants in the asymptotic notations depend on c.

Consider the sets $Z_2, \dots, Z_l$. These sets are subsets of R and $|Z_i| \ge \gamma \mathrm{Vol}(R)$ for some positive constant γ depending on c. Let g be a large constant integer. With the exception of at most $g-1$ sets, we partition the Z_i's into $l_1 = \lfloor (l-1)/g \rfloor$ disjoint groups of size g: $G_1, \dots, G_{l_1}$. Thus each group G_j contains g sets, each of which is a subset of R with cardinality $\gamma \mathrm{Vol}(R)$ for some positive constant γ. By setting g sufficiently large, the general filling lemma (Lemma 5.5) applies and shows that the sum of the sets in any group G_j contains a proper GAP Q_j of cardinality $\Omega(\mathrm{Vol}(R))$. Moreover, the rank of Q_j is the same as the rank of R and the differences of Q_j are multiples of the differences of R.

Since $|Q_j| = \Omega(\mathrm{Vol}(R))$, there are only $O(1)$ choices for the difference set of Q_j (for the definition of difference sets, see Section 3). Thus, a constant fraction of the Q_j's has the same difference set. Without loss of generality, we may assume that these Q_j's are $Q_1, Q_2, \dots, Q_{l_2}$, where $l_2 = \Omega(l_1)$.

Since $|Q_j| = \Omega(\mathrm{Vol}(R))$, the length of the hth edge of Q_j is $\Omega(1)$ times the length of the corresponding edge of R, for all $1 \le h \le \mathrm{rank}(R)$. Thus the lengths of the hth edge of the Q_j's are within a constant factor from each other, for all $1 \le k \le \mathrm{rank}(R)$. This implies that the intersection of the boxes $B_{Q_1}, \dots, B_{Q_{l_2}}$ contains a box B with volume $\Omega(\mathrm{Vol}(R))$ (for the definition of these boxes, see subsection 3.13). Let $m_1, \dots, m_d$ be the lengths of the edges of B, and let $(a_1, \dots, a_d)$ be the (common) set of differences of $Q_1, \dots, Q_{l_2}$. It follows that each of $Q_1, \dots, Q_{l_2}$ contains a translation of the proper GAP $Q = \{a_1x_1 + \dots + a_dx_d | 0 \le x_i \le m_i\}$ (Q is proper because the Q_j's are proper). We have that

$$|Q| = |B| = \Omega(\mathrm{Vol}(R)) = \Omega(|X|)$$

and

$$l_2 = \Omega(l_1) = \Omega(l).$$

Moreover, a translation of l_2Q is contained in $Q_1 + \dots + Q_{l_2}$, and a translation of $Q_1 + \dots + Q_{l_2}$ is contained in $X_1 + \dots + X_l$. So $X_1 + \dots + X_l$ contains a translation of l_2Q, completing the proof. □

5.9. **Proof of Theorem 5.1.** With the main lemma in hand, we are ready to conclude the proof of Theorem 5.1. In order to find a triplet (A', l', n') as desired, we are going to apply the so-called tree argument. This argument was introduced in [28] and, in spirit, works as follows. Assume that we want to add several sets $A_1, \dots, A_l$. We shall add them in a special way following an algorithm which assigns sets to the vertices of a tree. A set of any vertex contains the sum of the sets of its children. If the set at the root of the tree is not too large, then there is a level where the sizes of the sets do not increase (compared with the sizes of their children) too much. Thus, we can apply Freiman's inverse theorems at this level to deduce useful information. The creative part of this argument is to come up with a proper algorithm which suits our need.

The reader has already seen a simple version of this argument in the proof of Theorem 3.8. In that proof, the sets at the leaves of the tree are copies of A, the sets at a level i are copies of $2^i A$, and the set at the root is lA. A set of any vertex is the sum of the sets at its two children.

The algorithm in the current case is more complicated. Before describing it, let us assume, without loss of generality, that l is a power of 4 ($l = 4^s$) and $|A_i| = n_1$ and $0 \in A_i$ for all $1 \leq i \leq l$. Set $A_i = A_1^1$ for $i = 1, \dots, l$ and $l_1 = l$. Here is the description of the algorithm.

The algorithm. At the tth step, the input is a sequence $A_1^t, \dots, A_{l_t}^t$ of the same cardinality n_t where l_t is an even number. Choose a pair $1 \leq i < j \leq l_t$ which maximizes $|A_i^t + A_j^t|$ (if there are many such pairs, choose an arbitrary one). Denote the sum $A_i^t + A_j^t$ by A_1'. Remove i and j from the index set and repeat the operation to obtain A_2' and so on. After $l_t/2$ operations we obtain a sequence $A_1', \dots, A_{l_t/2}'$ of sets with decreasing cardinalities. Define $l_{t+1} = l_t/4$. Consider the sequence $A_1', \dots, A_{l_{t+1}}'$, and truncate all but the last set so that all of them have the same cardinality (which is $|A_{l_{t+1}}'|$). The truncated sets will be named $A_1^{t+1}, \dots, A_{l_{t+1}}^{t+1}$, and they form the input of the next step. It is clear that $l_t = \frac{l_1}{4^{t-1}}$ for all plausible t's. The algorithm halts at time $s+1$ where $l_{s+1} = 1$.

Notice that $A_{l_{t+1}}^{t+1}$ is a subset of $[2^t n]$, so $n_{t+1} \leq 2^t n$. We first show that there is some $t \leq s$ so that $n_{t+1} \leq 4^{d+1} n_t$. Assume otherwise. Then

$$n_{s+1} \geq (4^{d+1})^s n_1 = (4^s)^d 4^s |A| = 4^s l^d |A| > 2^s n,$$

a contradiction. In the following, let t be the first index so that $n_{t+1} \leq 4^{d+1} n_t$. By the description of the algorithm, there are $l_t/2$ sets among the sets A_i^t's such that every pair of them has cardinality at most $n_{t+1} \leq 4^{d+1} n_t$. Let us call these sets $B_1, \dots, B_{l_t/2}$. We have

- $|B_1| = \cdots = B_{l_t/2} = n_t$,
- B_i's are subsets of the interval $[2^{t-1} n]$,
- $|B_i + B_j| \leq 4^{d+1} n_t$, for all $1 \leq i < j \leq l_t/2$.

By Lemma 5.8, the sum $B_1 + \cdots + B_{l_t/2}$ contains a translation of $l'A'$, where $l' \geq \epsilon l_t/2$ and A' is a proper GAP with cardinality at least ϵn_t and ϵ is a positive constant depending on d. Moreover, A' is a subset of $[k_1 2^{t-1} n]$, for some constant k_1 depending on d. Set $n' = k_1 2^{t-1} n$. To conclude the proof, let us verify that l', n', and A' satisfy the required relations. First of all

$$(l')^d |A'| \geq (\epsilon l_t/2)^d \epsilon n_t \geq \frac{\epsilon^{d+1}}{2^d} \frac{l^d}{4^{(t-1)d}} 4^{(d+1)(t-1)} |A| \geq \frac{\epsilon^{d+1}}{2^d} 2^{t-1} l^d |A|. \tag{20}$$

Since $l^d |A| \geq Cn$, it follows that

$$(l')^d |A'| \geq \frac{C\epsilon^{d+1}}{2^d} 2^{t-1} n = \frac{C\epsilon^{d+1}}{k_1 2^d} n'.$$

By increasing C (notice that ϵ and k_1 do not depend on C), we can assume that $(l')^d |A'|/n'$ is sufficiently large. This guarantees that the condition of Theorem 3.12

is met. Replacing d by d' in (20), one can verify that for any $d' \leq d$

$$(l')^{d'}|A'| = \Omega(l^{d'}|A|),$$

This concludes the proof. □

6. Folkman's conjecture on subcomplete sequences

For a (finite or infinite) set A, S_A denotes the collection of subset sums of A:

$$S_A = \{\sum_{x \in B} x | B \subset A, |B| < \infty\}.$$

An infinite sequence A of positive integers is *subcomplete* if S_A contains an infinite arithmetic progression. Subcomplete sequences have been studied extensively, and we refer the reader to Section 6 of the monograph [9] by Erdös and Graham for a survey. For an infinite sequence A, we use $A(n)$ to denote the number of elements of A between 1 and n. This number could be larger than n as A might contain the same number many times. In 1966, Folkman made the following conjecture.

Conjecture 6.1. *There is a constant C such that the following holds. If $A = \{a_1 \leq a_2 \leq a_3 \leq \dots\}$ is an infinite non-decreasing sequence of positive integers and $A(n) \geq Cn$, for all sufficiently large n, then A is subcomplete.*

Folkman's conjecture was considered by Erdös and Graham as the most important conjecture concerning subcomplete sequences ([9], Section 6). Folkman himself proved that the conjecture holds under the stronger condition that $A(n) \geq n^{1+\epsilon}$, where ϵ is an arbitrary positive constant. The conjecture is sharp, as one cannot replace n by $n^{1-\epsilon}$. To show this, let us present an observation of Erdös [8].

Fact 6.2. Consider an infinite sequence $A = \{a_1, a_2, \dots\}$. If

$$\limsup_{i \to \infty} \left(a_i - \sum_{j=1}^{i-1} a_j \right) \to \infty, \tag{21}$$

then A is not subcomplete.

To verify Fact 6.2, notice that if A is subcomplete and d is the difference of an infinite arithmetic progression contained in S_A, then d is lower bounded by $\limsup_{i \to \infty} (a_i - \sum_{j=1}^{i-1} a_j)$.

For any fixed $\epsilon > 0$, it is simple to find a non-decreasing sequence A such that $A(n) = \Omega(n^{1-\epsilon})$ and A satisfies (21).

Using a special case of Theorem 5.1, we are able to confirm Folkman's conjecture.

Theorem 6.3. *There is a constant C such that the following holds. If $A = \{a_1 \leq a_2 \leq a_3 \leq \dots\}$ is an infinite non-decreasing sequence of positive integers and $A(n) \geq Cn$, for all sufficiently large n, then A is subcomplete.*

The rest of this section is devoted to the proof of Theorem 6.3, which relies on Corollary 5.2. First, we prove a sufficient condition for subcompleteness. This condition is of independent interest and will be used for another problem in Section 9. To complete the proof, we show that any sufficiently dense sequence satisfies this condition. This part of the proof makes significant use of Corollary 5.2.

6.4. **A sufficient condition for subcompleteness.** We say that a sequence A admits a *good* partition if it can be partitioned into two subsequences A' and $A^{''}$ with the following two properties:

- There is a number d such that $S_{A'}$ contains an arbitrarily long arithmetic progression with difference d.
- Let $A^{''} = b_1 \le b_2 \le b_3 \le \dots$. For any number K, there is an index $i(K)$ such that $\sum_{j=1}^{i-1} b_j \ge b_i + K$ for all $i \ge i(K)$.

Admitting a good partition is a sufficient condition for subcompleteness.

Lemma 6.5. *Any sequence A which admits a good partition is subcomplete.*

Proof of Lemma 6.5. We start with a definition.

Definition 6.6. An infinite sequence $B = \{b_1 \le b_2 \le b_3 \le \dots\}$ is a (d, L)-net if $b_{i+1} - b_i < L$ and is divisible by d for all $i = 1, 2 \dots$.

Observe that if B is a (d, L)-net and Q is a finite arithmetic progression with difference d and length greater than L/d, then $B + Q$ contains an infinite arithmetic progression with difference d. This observation is the leading idea in what follows.

Assume that A admits a good partition, and let $Q_0, Q_1, Q_2, \dots$ be arithmetic progressions with the same difference d and strictly increasing lengths contained in $S_{A'}$. The existence of the Q_i's is guaranteed by the first property of a good partition.

Next, we focus on $A^{''}$. Let X be the set of divisors d' of d with the following property: All but at most a finite number of elements of $A^{''}$ are divisible by d'. Since $1 \in X$, X is not empty and thus has a maximum element d_1. By throwing away a finite number of elements, we can assume that all elements of $A^{''}$ are divisible by d_1. Next, discard all elements y (in the remaining sequence) if there is only a finite number of elements of $A^{''}$ which equal y modulo d. Again, we discard only a finite number of elements so the remaining sequence still has the same density as $A^{''}$. Thus, we can assume that $A^{''} = \{b_1 d_1 \le b_2 d_1 \le \dots\}$ where the b_i's have the following property: Let b'_i be the remainder when dividing b_i by d. For each i, there are infinitely many j's such that $b'_i = b'_j$. Moreover, the greatest common divisor of the b'_i's equals 1 modulo d by the definition of d_1. We next need the following elementary fact, which is a consequence of the Chinese remainder theorem.

Fact 6.7. Let $1 \le z_1 \le z_2 \le \dots \le z_h < d$ be positive integers. If $\gcd(z_1, \dots, z_h) = 1(\bmod\ d)$, then there are integers $0 \le a_1, \dots, a_h < d$ such that $\sum_{j=1}^{h} a_j z_j \equiv 1(\bmod\ d)$.

By Fact 6.7 and the property of $A^{''}$ described in the previous paragraph, we can find $(d-1)$ mutually disjoint finite subsets $X_1, \dots, X_{d-1}$ of $A^{''}$ so that the sum of the elements in each subset equals d_1 modulo d. Denote these sums by $x_1 d + d_1, \dots, x_{d-1} d + d_1$, where the x_i's are non-negative integers. For any arithmetic progression Q_j with length $l \ge 3(x_1 + \dots + x_{d-1})$, the set $Q_j + S_{\{x_1 d + d_1, \dots, x_{d-1} d + d_1\}}$ contains an arithmetic progression with difference d_1 and length at least $l/2$ (recall that Q_j has difference d which is divisible by d_1). Since the lengths of the Q_j's go to infinity with j, we can conclude that $S_{A'} + S_{\{x_1 d + d_1, \dots, x_{d-1} d + d_1\}}$ contains an arbitrarily long arithmetic progression with difference d_1.

Set $A^{'''} = A^{''} \backslash \bigcup_{i=1}^{d-1} X_i$; to complete the proof of the lemma, it suffices to prove that $S_{A^{'''}}$ contains a (d_1, L)-net for some constant L. Let $S_{A^{'''}} = \{s_1 < s_2 < \dots\}$.

Every elements of $A^{''}$ is divisible by d_1 and so are all s_i's. Therefore, it suffices to exhibit the existence of a constant L satisfying $s_{i+1} - s_i \leq L$ for all i. The existence of L follows directly from the following observation, due to Graham [15], and the second property of a good partition (this is the only place where we use this property).

Fact 6.8. Let $Y = y_1 < y_2 < \dots$ be an infinite sequence of positive integers, and let $S_Y = \{s_1 < s_2 < \dots\}$. If $y_{m+1} \leq \sum_{i=1}^m y_i$ for all sufficiently large m, then there is some L such that $s_{i+1} - s_i \leq L$ for all i.

Proving Fact 6.8 is not too hard, and the reader might want to consider it as an exercise. □

6.9. **Proof of Theorem 6.3.** We first present a lemma which provides a link between good partitions and subcompleteness. This lemma is a simple, but a bit tricky, consequence of Corollary 5.2.

Lemma 6.10. *There is a constant C such that the following holds. If A is a multi-set of positive integers between* 1 *and n and $|A| \geq Cn$, then S_A contains an arithmetic progression of length n.*

Proof of Lemma 6.10. We show that the same constant C in Corollary 5.2 suffices. Without loss of generality, we assume that C is an integer and $|A| = Cn$. If the multi-set A contains an element a of multiplicity n, then the arithmetic progression $a, 2a, \dots, na$ is a subset of S_A and we are done. In the other case, we can partition the Cn elements of A into n sets $X_1, \dots, X_n$ such that each X_i consists of exactly C different elements. The sum $X_1 + \dots + X_n$ is a subset of S_A. Corollary 5.2 implies that the sum $X_1 + \dots + X_n$ contains an arithmetic progression of length n, given that C is sufficiently large. This concludes the proof of the lemma. □

With Lemma 6.10 in hand, we are in a position to prove that the sequence A in Theorem 6.3 admits a good partition, provided that the constant C in this theorem is sufficiently large. The partition is the most natural one. Assume that the elements of A are ordered non-decreasingly as $A = a_1 \leq a_2 \leq a_3 \leq \dots$; A' consists of the elements with odd indices, and $A^{''}$ consists of those with even indices.

By definition, $A^{''} = \{a_2, a_4, a_6, \dots\}$. Since $A(n) \geq Cn$ for all sufficiently large n (recall that $A(n)$ is the number of elements of A between 1 and n), for every sufficiently large even number j,

$$a_j \leq j/C \leq j/5 \leq a_2 + a_4 + \dots + a_{j-2} - j/4,$$

which guarantees the property required for $A^{''}$.

It remains to check the property concerning A'. As A has density Cm, A' has density $Cm/2$, so we can assume that $A' = \{b_1 \leq b_2 \leq \dots\}$, where $b_m \leq 2m/C$ for all sufficiently large m. Let $A'[m]$ be the set consisting of the first m elements of A'. Fix a sufficiently large m, and define $A_0 = A'[m]$ and $A_i = A'[2^i m] \backslash A'[2^{i-1}m]$. The set A_i has $2^{i-1}m$ elements and is a subset of the interval $[2^{i+1}m/C]$.

To conclude the proof, we make use of the following lemma, proved in [27]:

Lemma 6.11. *Let P be a generalized arithmetic progression of rank* 2*, $P = \{x_1a_1 + x_2a_2 | 0 \leq x_i \leq l_i\}$, where $l_i \geq 5a_{3-i}$ for $i = 1, 2$. Then P contains an arithmetic progression of length $l_1 + l_2$ whose difference is* $\gcd(a_1, a_2)$.

By Lemma 6.10 (provided that C is sufficiently large), S_{A_i} contains an arithmetic progression P_i of length $l_i = 2^{i+1}m/C$ for all i. Set $Q_0 = P_0$ (and assume that d_0 is the difference of Q_0), and consider the generalized arithmetic progression $Q_0 + P_1$. This is a generalized arithmetic progression of rank 2 with volume $l_0 l_1$. Moreover, this two dimensional generalized arithmetic progression is a subset of an interval of small length, so one can easily check that its differences are relatively small and satisfy the assumption of Lemma 6.11. This lemma implies that $Q_0 + P_1 = P_0 + P_1$ contains an arithmetic progression Q_1 of length $l_0 + l_1 - 2$ with difference d_1 which is a divisor of d_0. (The -2 term comes from the fact that in Lemma 6.11, the edges of P have length $l_1 + 1$ and $l_2 + 1$, respectively; this term, of course, plays no role.) Similarly, by considering $Q_1 + P_2$, we obtain an arithmetic progression Q_2 of length $l_0 + l_1 + l_2 - 3$ with difference d_2 which is a divisor of d_1 and so on. The sequence $d_0, d_1, d_2, \dots$ is non-increasing, so there is an index j so that $d_i = d_j = d$ for all $i \geq j$. The arithmetic progressions $Q_j, Q_{j+1}, Q_{j+2}, \dots$ have strictly increasing lengths and the same difference d. Moreover, each Q_i is a subset of $S_{A'}$, and this completes the proof. □

7. Sumsets with distinct summands

In this section, we strengthen Theorem 3.12 in another direction. Instead of the sumset lA, we are going to consider the much more restricted sumset l^*A, which consists of the sums $a_1 + \cdots + a_l$ where the a_i's are different elements of A.

Theorem 7.1. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. For any positive integers n and l and any set $A \subset [n]$ satisfying $l \leq |A|/2$ and $l^d|A| \geq Cn$, l^*A contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer $1 \leq d' \leq d$.*

The requirement that the summands must be different usually poses a great challenge in additive problems. One of the most well-known examples is the celebrated Erdös-Heilbronn's conjecture. In order to describe this conjecture, let us start with the classical Cauchy-Davenport theorem which asserts that if A is a set of residues modulo n, where n is a prime, then $|2A| \geq \min\{n, 2|A| - 1\}$. For A an arithmetic progression, the bound is sharp. Now let us consider 2^*A. We want to bound $|2^*A|$ from below with something similar to Cauchy-Davenport's bound. Observe that in the special case when A is an arithmetic progression, $|2^*A| = \min\{n, 2|A| - 3\}$. Thus one may guess that

$$|2^*A| \geq \min\{n, 2|A| - 3\} \tag{22}$$

holds for any set A. This is what Erdös and Heilbronn conjectured. While Cauchy-Davenport's theorem is quite easy to prove, Erdös-Heilbronn's conjecture had been open for about thirty years until it was solved by da Silva and Hamidoune in 1994 [7].

It is now not so big a surprise that Theorem 7.1 is harder and deeper than both Theorem 3.12 and Theorem 5.1. The proof of Theorem 7.1 uses Theorem 3.12 as a lemma and requires lots of additional arguments, but let us take a gentle start by introducing some simple ideas.

7.2. **The initial ideas.** The initial ideas in the proof of Theorem 7.1 are similar to those in the proof of Theorem 5.1. We want to show that there are numbers

l', n' and a set A' such that

- A' is a subset of $[n']$ and $l', n', |A'|$ satisfy the conditions of Theorem 3.12, namely $l'^d|A'|/n'$ is sufficiently large;
- $(l')^{d'}|A'| = \Omega(l^{d'}|A|)$ for all $1 \leq d' \leq d$.

In the rest of the proof, we call a triple (A', l', n') *perfect* if it satisfies the above two conditions. If we could show that there is a perfect triple (A', l', n') such that $l'A'$ is a subset of l^*A, then an application of Theorem 3.12 to this triple immediately implies the statement of Theorem 7.1.

It is useful to notice that in Theorem 7.1, instead of the assumption $l \leq |A|/2$, we can afford the stronger assumption that $l \leq \epsilon|A|$ for any positive constant ϵ, at the cost of increasing the constant C. One can argue as follows. First one puts aside $(1-\epsilon)l$ elements from A. Next, consider the pair (A_1, l_1) where A_1 is the set of the remaining $|A| - (1-\epsilon)l$ elements and $l_1 = \epsilon l$. It is trivial that $l_1 \leq \epsilon|A_1|$. On the other hand, the sum of an element from $l_1^*A_1$ and of the $(1-\epsilon)l$ elements put aside is an element of l^*A. So if $l_1^*A_1$ contains a proper GAP P, then l^*A contains a translation of P.

The above argument also shows that for any $l_1 < l$, if A_1 is a subset of at most $|A| - (l - l_1)$ elements of A, then $l_1^*A_1$ is a subset of a translation of l^*A.

In the proof of Theorem 7.1, we shall assume that $l \leq \epsilon|A|$, whenever needed. We shall also assume that $l-1$ elements of A are put aside in case we need them to create the sum of exactly l elements. These assumptions provide us some flexibility in constructing a perfect triple. In particular, we shall not need to show that $l'A'$ is a subset of l^*A; it suffices to show that $l'A'$ is a subset of a translation of $\tilde{l}^*A$, for some $\tilde{l} \leq l$.

The main part of the proof is to construct a perfect triple, and this is significantly harder than what we did in the proof of Theorem 5.1. However, when $|A|$ is large, the construction is relatively simple, and we start with this case. The treatment of the harder case when $|A|$ is relatively small starts in subsection 7.5, where we present a key structural lemma. The proof of this lemma occupies the rest of this section. In Section 8, we present the rest of the proof of Theorem 7.1.

7.3. **The case when $|A|$ is large.** Let A_1 be a subset of A with cardinality $l-1$, and set $A_2 = A \backslash A_1$. Since $|A| \geq 2l$,

$$|A_2| \geq \frac{|A|}{2}. \tag{23}$$

We assume, with foresight (and with room to spare), that $|A|^2 \geq 80Cn\log_2 n$ and $l^d|A| \geq 160 \times 2^dCn$, where C is the constant in Theorem 3.12.

Define $m_i = 2^i$ for all $1 \leq i \leq t$, where t is the smallest index such that $m_t \geq |A_2|/2$. Since $|A_2| \leq |A| \leq n$, $t \leq \log_2 n$. Let S_i be the set of those numbers in $[2n]$ which can be represented as the sum of two different elements in A_2 in at least m_i and less than m_{i+1} ways. It is essential to observe that m_iS_i is a subset of $(2m_i)^*A$. On the other hand, a simple double counting argument gives

$$\sum_{i=1}^{t} m_i|S_i| \geq \binom{|A_2|}{2} - 4n \geq q = \frac{|A|^2}{5}. \tag{24}$$

Next, we split $\sum_{i=1}^t m_i|S_i|$ into three parts. The first part comprises those $m_i|S_i|$ where $m_i|S_i| \leq \frac{q}{4t}$. Obviously, the contribution of this part to the sum is at most

$t\frac{q}{4t} = \frac{q}{4}$. The second part consists of those $m_i|S_i|$ where $|S_i| \leq \frac{|A_2|}{40}$. Since the sequence m_i is geometric, the sum of all m_i's is bounded from above by $2|A_2|$. Thus, the contribution of the second part is upper bounded by $2|A_2|\frac{|A_2|}{40} \leq \frac{q}{4}$. The third part contains the remaining $m_i|S_i|$'s and, as a consequence of the previous estimates, its contribution is at least $\frac{q}{2}$.

Let $i_1 < i_2 < \cdots < i_j$ be the indices in the third part. We have

$$\sum_{g=1}^{j} m_{i_g}|S_{i_g}| \geq \frac{q}{2}. \tag{25}$$

We are going to consider two cases:

(I) $2m_{i_j} > l$: In this case $|S_{i_j}| \geq \frac{|A_2|}{40} \geq \frac{|A|}{80}$, and $\frac{l}{2}S_{i_j}$ is a subset of l^*A. In view of the initial ideas presented in the previous subsection, we set $A' = S_{i_j}$, $n' = 2n$, and $l' = l/2$. Since $l^d|A| \geq 160 \times 2^d Cn$,

$$l'^d|A'| \geq (l/2)^d\frac{|A|}{80} \geq \frac{1}{80 \times 2^d}l^d|A| \geq 2Cn = Cn'$$

and

$$l'^{d'}|A| = \Omega(l^{d'}|A|),$$

for any $1 \leq d' \leq d$. The last two estimates guarantee that the triple (A', n', l') is perfect, and we are done.

(II) $2m_{i_j} \leq l$: In this case, we prove that l^*A contains an arithmetic progression of length $cl|A|$ (in other words, one can set the parameter d' in Theorem 7.1 equal to one). For any integer a which is the sum of $l - 2m_{i_j}$ different elements in A_1 (the set we put aside at the beginning of the proof), $a + m_{i_j}S_{i_j}$ is a subset of l^*A. On the other hand, as $|A|^2 \geq 80n\log_2 n$,

$$m_{i_j}|S_{i_j}| \geq \frac{q}{4t} \geq \frac{|A|^2}{20\log_2 n} \geq Cn.$$

Theorem 3.12 implies that $m_{i_j}S_{i_g}$ contains an arithmetic progression of length

$$cm_{i_j}|S_{i_j}| \geq c\frac{q}{4t} \geq c\frac{|A|^2}{20\log_2 n} \geq cl|A|$$

if $l \leq |A|/20\log_2 n$. The case when l is larger than $|A|/20\log_2 n$ requires an extra argument. Notice that by the definition of the third partial sum and the assumption on $|A|$

$$\frac{1}{2}m_{i_g}|S_{i_g}| \geq \frac{1}{2}\frac{q}{4t} \geq \frac{|A|^2}{40\log_2 n} \geq Cn.$$

Given this, we can apply Theorem 3.12 to $\frac{1}{2}m_{i_g}S_{i_g}$ to obtain an arithmetic progression of length $cm_{i_g}|S_{i_g}|$, for every index g in the third partial sum. To conclude, we use the following simple fact to glue these arithmetic progressions together.

Fact 7.4. Any element in $\sum_{g=1}^{j}\frac{1}{2}m_{i_g}S_{i_g}$ can be represented by the sum of $m = \sum_{g=1}^{j} m_{i_g}$ different elements from A_2.

Proof of Fact 7.4. Greedy algorithm. □

It follows that $\sum_{g=1}^{j} \frac{1}{2} m_{i_g} S_{i_g}$ is a subset of $m^* A_2$, with m defined as in Fact 7.4. Finally, by applying Lemma 6.11 iteratively, one can show that $\sum_{g=1}^{j} \frac{1}{2} m_{i_g} S_{i_g}$ contains an arithmetic progression of length

$$c \sum_{g=1}^{j} m_{i_g} |S_{i_g}| \geq c\frac{q}{2} \geq c\frac{|A|^2}{10} \geq c\frac{l|A|}{5}.$$

Now we can add additional elements from A_1 to $m^* A_2$ to obtain a subset of $l^* A$. □

This simple proof, unfortunately, cannot be repeated for the case $|A| = o(\sqrt{n})$. However, the arguments presented here will be useful later on.

7.5. **A structural lemma.** In view of the result in the previous subsection, we only have to deal with the case $|A| = O(\sqrt{n \log n})$. Actually, this upper bound on $|A|$ matters little, but it imposes a bound on l that is critical. Notice that if $|A| = O(\sqrt{n \log n})$, then in order to guarantee the assumption $l^d |A| \geq Cn$ of Theorem 7.1, we must have

$$l = \Omega(n^{1/2d - o(1)}) \gg \log_2^{10} n.$$

In this subsection, we focus on those pairs (l, A), where $l^d |A|$ is close to n (but not necessarily larger than n) and l is relatively large. A key step in our proof is the following structural lemma, which asserts that if $l^* A$ does not yield a proper GAP as claimed by Theorem 7.1, then A must contain a big subset which has a very rigid structure.

Lemma 7.6. *For any positive constants ν and d there are positive constants δ, α, and d_1 such that the following holds. Let A be a subset of $[n]$, let l be a positive integer, and let $n \geq f(n) \geq 1$ be a function of n such that*

$$\max\{\log^{10} n, (40 f(n) \log_2 n)^{1/3d}\} \leq l \leq |A|/2$$

and $l^d |A| f(n) \geq n$. Then one of the following two statements must hold:

- *$l^* A$ contains a proper GAP of rank d' and volume $\Omega(l^{d'} |A|)$ for some $1 \leq d' \leq d$.*
- *There is a subset $\tilde{A}$ of A with cardinality at least $\delta |A|$ which is contained in a GAP P of rank d_1 and volume $O(|A| f(n)^{1+\nu} \log^{\alpha} n)$.*

The function $f(n)$ can be seen as a *rigidity* parameter. The closer $l^d |A|$ is to n, the more rigid is the structure of $\tilde{A}$. With some extra work, the lower bound of l in the lemma can be improved: 10 can be replaced by any constant larger than 1, and $1/3d$ can be replaced by any positive constant. If we refine the result this way, the constants α, ν, and d_1 will also depend on the new constants.

For the proof of Theorem 7.1, we only need the special case when $f(n) = 1$. We, however, choose to present Lemma 7.6 in the above general form since it might be of independent interest, and the proof is not significantly harder than that of the special case.

With $f(n) = 1$, Lemma 7.6 yields the following corollary.

Corollary 7.7. *For any positive constant d there are positive constants δ, α and d_1 such that the following holds. Let A be a subset of $[n]$, and let l be a positive integer such that $l^d |A| \geq Cn$. Then one of the following two statements must hold.*

- *l^*A contains a proper GAP of rank d' and volume $\Omega(l^{d'}|A|)$ for some $1 \leq d' \leq d$.*
- *There is a subset $\tilde{A}$ of A with cardinality at least $\delta|A|$ which is contained in a GAP P of rank d_1 and volume $O(|A| \log^\alpha n)$.*

Notice that the set $\tilde{A}$ in Corollary 7.7 satisfies

$$l^d|\tilde{A}| \geq l^d \delta |A| \geq \delta C n.$$

Since δ depends only on d, by increasing the constant C in Theorem 7.1, we can always assume that δC is sufficiently large. Thus, given Corollary 7.7, it suffices to prove Theorem 7.1 under the additional condition that A be a subset of density at least $\frac{1}{\log^\alpha n}$ of a GAP of constant rank, where both the rank and α are constants depending on d. We present this proof in the next section. A reader who is eager to see this proof can delay the reading of the rest of this section and jump right to Section 8.

The rest of this section is devoted to the proof of Lemma 7.6. As this proof is fairly long, we brake it into four parts, each of which contains arguments of a fairly different nature. The main technical ingredient of this proof is again a tree argument, similar to what we used in the proof of Theorem 5.1. However, the algorithm here is more complicated than the algorithm in Section 5, and the analysis is also more challenging.

In order to set up the algorithm, we first need to produce a great amount of subsets of A with a certain property. This will be done in the next subsection. In subsection 7.10, we describe our algorithm together with several simple observations. Subsection 7.12 is devoted to an inverse argument, which we use to derive the desired properties of A. This derivation is quite different from and much more tricky than the one in Section 5. We wrap up with the final subsection, subsection 7.14, which contains the verification of an estimate claimed in subsection 7.12.

7.8. **Small sets with big sums.** The goal of this subsection is to show that any finite set A contains a subset B of small size ($O(\ln |A|)$) such that $|l^*B|$ is large, where $l = |B|/2$.

Lemma 7.9. *Let A be a finite set of real numbers where $|A|$ is sufficiently large. Then A contains a subset B of at most $20 \log_2 |A|$ elements such that $(\frac{|B|}{2})^*B$ has cardinality at least $|A|$.*

Proof of Lemma 7.9. We can assume, without loss of generality, that $|A|$ is sufficiently large so that $|A| \geq 100 \log_2 |A|$. We choose the first two elements of A, say a_1, a_2, arbitrarily. Once $a_1, \dots, a_{2i}$ have been chosen, we next choose a_{2i+1} and a_{2i+2} from $A \backslash \{a_1, \dots, a_{2i}\}$ such that

$$|(i+1)^*\{a_1, \dots, a_{2i+1}, a_{2i+2}\}| \geq 1.1 |i^*\{a_1, \dots, a_{2i}\}| \tag{26}$$

(if there are many possible pairs, we choose an arbitrary one). We stop at time T when $|T^*\{a_1, \dots, a_{2T}\}| \geq |A|$, and we let $B = \{a_1, \dots, a_{2T}\}$. It is clear that $|B| \leq 2 \log_{1.1} |A| \leq 20 \log_2 |A|$. The only point we need to make now is to show that as far as $|i^*\{a_1, \dots, a_{2i}\}| < |A|$, we can always find a pair (a_{2i+1}, a_{2i+2}) to satisfy (26). Assume (for contradiction) that we get stuck at the ith step, and denote by S the sumset $i^*\{a_1, \dots, a_{2i}\}$. For any two numbers $a, a' \in A \backslash \{a_1, \dots, a_{2i}\}$,

$(a+S)\cup(a'+S)$ is a subset of $(i+1)^*\{a_1,\dots,a_{2i},a,a'\}$. So by the assumption, we have

$$|(a+S)\cup(a'+S)|\le 1.1|S|.$$

Since both $a+S$ and $a'+S$ have $|S|$ elements, it follows that their intersection has at least $.9|S|$ elements. This implies that the equation $a'-a=x-y$ has at least $.9|S|$ solutions (x,y) where $x\in S$ and $y\in S$. Now let us fix a as the smallest element of $A\backslash\{a_1,\dots,a_{2i}\}$ and choose a' arbitrarily. There are $|A|-2i-1\ge .9|A|$ choices for a', each of which generates at least $.9|S|$ pairs (x,y) where both x and y are elements of S. As all (x,y) pairs are different, we have that

$$.9|A|\times .9|S|\le\binom{|S|}{2},$$

which implies that $|S|>|A|$, a contradiction. This concludes the proof. □

Many good small sets. Consider a set A as in Theorem 7.1. Apply Lemma 7.9 to A to obtain a small set A_1. Next, apply the lemma to $A\backslash A_1$ to obtain a small set A_2 and so on. Each time we add to A_i a few "dummy" elements to make its cardinality exactly $20\log_2|A|$. Stop when $A\backslash(\bigcup_{i=1}^m A_i)$ has less than $2|A|/3$ elements for the first time. Without loss of generality, we can assume that $20\log_2|A|$ is even and set $l_0=10\log_2|A|$. We have a collection $A_1,\dots,A_m$ of disjoint subsets of A with the following properties.

- $|A_1|=\cdots=|A_m|=20\log_2|A|=2l_0$.
- $|l_0^*A_i|=\left|\frac{|A_i|}{2}^* A_i\right|\ge(2/3-o(1))|A|>|A|/2$.
- $|A\backslash(\bigcup_{i=1}^m A_i)|=(2/3+o(1))|A|$.

Here we assume that $\log_2|A|=o(|A|)$, which explains the error terms $o(1)$ in the last two properties. In the next subsection, we consider an algorithm which uses the sets A_i as input.

7.10. **The algorithm.** Set $B_i=l_0^*A_i$ for all $1\le i\le m$. We now give a description of our algorithm. This algorithm constructs a subset of l^*A in a particular way. We shall exploit the fact that the cardinality of this subset is at most $|l^*A|\le ln$ (since l^*A itself is a subset of $[ln]$) in order to derive information about A.

The algorithm. To start, set $m_0=m$. Truncate the set of B_i's so each of them has exactly $b_0=|A|/2$ elements. Denote by B_i^0 the truncation of B_i. We start with the sequence of sets $B_1^0,\dots,B_{m_0}^0$, each of which has exactly b_0 elements. Without loss of generality, we may assume that m_0 is a power of 4. At the beginning, we call the elements in $A^{[1]}=A\backslash(\bigcup_{i=1}^m A_i)$ *available.*

A general step of the algorithm functions as follows. The input is a sequence $B_1^t,\dots,B_{m_t}^t$ of sets of the same cardinality b_t. Consider the sets $\bigcup_{h=1}^K(B_i^t+B_j^t+x_h)$ where $1\le i<j\le m_t$ and $x_1,\dots,x_K$ are different available elements (K is a large constant to be specified later). Choose $i,j,x_1,\dots,x_K$ such that the cardinality of $B_1'=\bigcup_{h=1}^K(B_i^t+B_j^t+x_h)$ is maximum (if there are many possibilities, choose an arbitrary one). Remove i and j from the index set and the x_i's from the available set and repeat the operation to obtain B_2' and so on. We end up with a set sequence $B_1',\dots,B_{m_t/2}'$ where $|B_1'|\ge\cdots\ge|B_{m_t/2}'|$.

Let $m_{t+1}=m_t/4$, and set $b_{t+1}=|B_{m_{t+1}}'|$. Truncate the B_i''s ($i<m_{t+1}$) so that the remaining sets have exactly b_{t+1} elements each. Denote by B_i^{t+1} the remaining subset of B_i'. The sequence $B_1^{t+1},\dots,B_{m_{t+1}}^{t+1}$ is the output of the step.

If $m_{t+1} \geq 4$, then we continue with the next step. Otherwise, the algorithm terminates.

Let us pause for a moment and make a series of observations. All of these observations are easy to verify, so we omit their proofs.

- Define $l_{t+1} = 2l_t + 1$ for $t = 0, 1, 2 \dots$. Then B_i^t is a subset of $l_t^* A$ for any plausible t and i.
- As A is a subset of $[n]$, B_i^t is a subset of $[l_t n]$.
- For any plausible t, $b_{t+1} \geq 2b_t$.
- After each step, the length of the sequence shrinks by a factor 4.
- At the beginning we have $(2/3 - o(1))$ available elements. The number of elements x_i's used in the whole algorithm is $o(|A|)$, so at any step, there are always $(2/3 - o(1))|A|$ available elements.

Since $l \gg l_0 = O(\log_2 n)$, we can assume, without loss of generality, that l/l_0 is a power of two, $l/l_0 = 2^{s_2}$. Recall that $m_0 = m \approx \frac{1}{3}\frac{|A|}{2l_0}$ (m is slightly larger than $\frac{1}{3}\frac{|A|}{2l_0}$) and $|A| \geq 2l$. It follows that $l/l_0 \leq 4m_0$. As we assume m_0 is a power of 4, $m_0 = 4^{s_1}$, it follows that $2(s_1 + 1) \geq s_2$.

We set $K = 2^{c_1 d}$, where c_1 is a constant at least 9. We first claim that

$$(K/2)^{s_2/2} > 40 l^d f(n) \log n. \tag{27}$$

Indeed, observe that

$$(K/2)^{s_1/2} \geq (2^{9d+1}/2)^{s_2/2} = 2^{9ds_2/2}. \tag{28}$$

Recalling the definition of s_2, $2^{s_2} = l/l_0$. We assume that $l \geq \log_2^{10} n \geq l_0^9$, so $2^{s_2} \geq l^{8/9}$. It follows that

$$2^{9ds_2/2} \geq l^{4d} = l^d \times l^{3d} \geq 40 l^d f(n) \log n,$$

by the assumption on l.

We next prove the following fact.

Fact 7.11. There is an index $k \leq s_2/2$ such that $b_k \leq K^k b_0$.

Proof of Fact 7.11. By the second observation we have that $b_k \leq l_k n$ for any k. From the definition of l_t it is easy to prove (using induction) that

$$l_k \leq 2^k l_0 + 2^k \leq 2^{k+1} l_0.$$

It follows that $b_k \leq 2^{k+1} l_0 n$ for any k. Recall that $b_0 = |A|/2$ and $l_0 = 10 \log_2 |A|$. If $b_k > K^k b_0$, then we should have

$$K^k |A|/2 = K^k b_0 < b_k \leq 2^{k+1} l_0 n \leq 2^{k+1} n \times (10 \log_2 |A|),$$

which implies

$$(K/2)^k |A| < 40 n \log_2 |A| \leq 40 n \log_2 n.$$

On the other hand, (27) and the assumption $l^d |A| f(n) \geq n$ of Lemma 7.6 together imply

$$(K/2)^{s_2/2} |A| \geq 40 l^d |A| f(n) \log_2 n \geq 40 n \log_2 n,$$

which is a contradiction. The proof is thus complete. □

7.12. **The inverse argument.** Let k be the first index where $b_k \leq K^k b_0$. This means $|B^k_{m_k}| \leq K^k b_0$. By the description of the algorithm

$$B^k_{m_k} = \bigcup_{h=1}^{K} (B_i^{k-1} + B_j^{k-1} + x_h) \tag{29}$$

for some i, j, and x_h's. Given (29), we are going to exploit the bound $|B^k_{m_k}| \leq K^k b_0$ in many ways. First, this bound and the definition of k means that $|B_i^{k-1} + B_j^{k-1}|$ is relatively small, and so we can use Freiman's theorem to derive some facts about the sets B_i^{k-1} and B_j^{k-1}. Next, (29) and the bound on $|B^k_{m_k}|$ imply that there should be a significant overlap among the sets of $(B_i^{k-1} + B_j^{k-1} + x_h)$'s. Thus, there should be a correlation between the (available) elements x_h. This correlation eventually leads us to a structural property of the set of available elements. The set $\tilde{A}$ claimed in the lemma will be a subset of this set.

To start, notice that (29) implies

$$|B^k_{m_k}| \geq |B_i^{k-1} + B_j^{k-1}| \tag{30}$$

where $1 \leq i < j \leq m_{k-1}$ and both B_i^{k-1} and B_j^{k-1} have cardinality $b_{k-1} \geq K^{k-1} b_0$. The definition of k then implies that $|B^k_{m_k}| \leq K b_{k-1}$, so

$$|B_i^{k-1} + B_j^{k-1}| \leq K |B_i^{k-1}|. \tag{31}$$

Applying Freiman's theorem to (31), we could deduce that there is a generalized AP R with constant rank containing B_i^{k-1} and $\mathrm{Vol}(R) = O(|B_i^{k-1}|) = O(b_{k-1})$.

We say that two elements u and v of B_j^{k-1} are equivalent if their difference is in $R - R$. If u and v are not equivalent, then the sets $u + B_i^{k-1}$ and $v + B_i^{k-1}$ are disjoint, since B_i^{k-1} is a subset of R. By (31), the number of equivalent classes is at most K. Let us denote these classes by $C_1, \dots, C_K$, where some of the C_s's might be empty. We have $B_i^{k-1} \subset R$ and $B_j^{k-1} \subset \bigcup_{s=1}^K C_s$.

Let us now take a close look at (29). The assumption $|B^k_{m_k}| \leq K|B_i^{k-1}|$ and (29) imply that there must be a pair s_1, s_2 such that the intersection

$$(B_i^{k-1} + B_j^{k-1} + x_{s_1}) \cap (B_i^{k-1} + B_j^{k-1} + x_{s_2})$$

is not empty. Moreover, the set $\{x_1, \dots, x_K\}$ in (29) was chosen optimally. Thus, for any set of K available elements, there are two elements x and y such that

$$(B_i^{k-1} + B_j^{k-1} + x) \cap (B_i^{k-1} + B_j^{k-1} + y)$$

is not empty. This implies

$$x - y \in (B_i^{k-1} + B_j^{k-1}) - (B_i^{k-1} + B_j^{k-1}) \subset \bigcup_{1 \leq g,h \leq K} \Big((R + C_g) - (R + C_h) \Big). \tag{32}$$

Define a graph G on the set of available elements as follows: x and y are adjacent if and only if $x - y \in (B_i^{k-1} + B_j^{k-1}) - (B_i^{k-1} + B_j^{k-1})$. By the argument above, G does not contain an independent set of size K, so there should be a vertex x with degree at least $|V(G)|/K$. By (32), there is a pair (g, h) such that there are at least $|V(G)|/K^3$ elements y satisfying

$$x - y \in (R + C_g) - (R + C_h). \tag{33}$$

Both C_g and C_h are subsets of translations of R; so the set $\tilde{A}$ of the elements y satisfying (33) is a subset of a translation of $P = (R + R) - (R + R)$. Recall that at any step, the number of available elements is $(1 - o(1))|A_2|$. We have

$$|\tilde{A}| \geq (1 - o(1))|A_2|/K^3 = \Omega(|A_2|). \tag{34}$$

Let us summarize what we have obtained here. We have found a subset $\tilde{A}$ of A of density at least $(2/3 - o(1))/K^3 = \Omega(1)$ and a GAP P which contains $\tilde{A}$. In order to complete the proof of the lemma, it remains to bound the volume of P. We need to show that if the first statement of the lemma does not hold, then

$$\mathrm{Vol}(P) = O(|A|f(n)^{1+\nu} \log^\alpha n). \tag{35}$$

At this point, we know that

$$\mathrm{Vol}(P) = O(\mathrm{Vol}(R)) = O(b_k), \tag{36}$$

where $b_k \leq K^k b_0 = K^k|A|$. Unfortunately, we still do not know much about K^k. Our next task is to prove that if the first statement of the lemma does not hold, then

$$K^k = O(f(n)^{1+\nu} \log^\alpha n), \tag{37}$$

which implies (35).

In order to verify (37), we need to exploit the definition of the sets $B^k_{m_k}$ even more. Notice that when we define $B^k_{m_k}$, we choose i and j optimally. On the other hand, as $m_k = \frac{1}{4}m_{k-1}$, for any remaining index i, we have at least $m' = m_{k-1}/2$ choices for j. This means that there are m' sets $B^{k-1}_{j_1}, \dots, B^{k-1}_{j_{l_2}}$, all of the same cardinality b_{k-1}, such that

$$|B^{k-1}_i + B^{k-1}_{j_r}| \leq K b_{k-1} \tag{38}$$

for all $1 \leq r \leq m'$.

From now on, we work with the sets $B^{k-1}_{j_r}$, $1 \leq r \leq m'$. By considering equivalent classes (as in the paragraph following (31)), we can show that for each r, $B^{k-1}_{j_r}$ contains a subset D_r which is a subset of a translation of R and $|D_r| \geq |B^{k-1}_{j_r}|/K = \Omega(\mathrm{Vol}(R))$.

By Lemma 5.5, there is a constant g such that $D_1 + \cdots + D_g$ contains a GAP Q_1 with cardinality at least $\gamma\mathrm{Vol}(R)$ for some positive constant γ. Using the next g D_i's, we can create Q_2 and so on. At the end, we have $m^{''} = \lfloor m'/g \rfloor$ generalized AP $Q_1, \dots, Q_{m''}$. Each of these has rank $d_1 = \mathrm{rank}(R)$ (this parameter d_1 is irrelevant to the whole argument) and cardinality at least $\gamma\mathrm{Vol}(R)$. Moreover, they are subsets of translations of the GAP $R' = gR$ which also has volume $O(\mathrm{Vol}(R))$.

Consider a GAP Q_i. Due to its large volume (compared to the volume of R'), there are only $O(1)$ possibilities for its difference set. Thus, there is a positive constant γ_1 such that at least a γ_1 fraction of the Q_i's has the same difference set. Truncating if necessary, we can assume the corresponding sides of these Q_i's have the same length (the truncation could decrease the volumes by at most a constant factor). Since two GAPs with the same difference sets and corresponding sides having the same length are translations of each other, we conclude that there is a GAP Q (of rank d_1 and cardinality at least $\gamma\mathrm{Vol}(R)$) and an integer $m^{'''} = \Omega(m^{''})$ such that there are least $m^{'''}$ translations of Q among the Q_i's. Without loss of generality, we can assume that these translations are $Q_1, \dots, Q_{m'''}$. Before continuing, let us gather some facts about Q_i and $m^{'''}$.

- $|Q| = |Q_i| = \Omega(\mathrm{Vol}(R)) = \Omega(b_k) = \Omega(K^k b_0) \geq \beta K^k b_0$, for some positive constant β.
- $m^{'''} = \Omega(m^{''}) = \Omega(m^{'}) = \Omega(m_k) = \Omega(m_0/4^k) \geq \mu m_0/4^k$, for some positive constant μ.

To proceed further, we need the following fact, whose proof is delayed until the next subsection.

Fact 7.13. If $\left(\frac{K}{2\times 4^d}\right)^k \geq f(n)\log^{d+2} n$, then there is $\bar{l} \leq l$ such that $(\bar{l})^*A$ contains a proper GAP of rank d' and volume $\Omega(l^{d'}|A|)$ for some $1 \leq d' \leq d$.

In order to have l^*A instead of $(\bar{l})^*A$, one can do the usual "reserving" trick. Prior to Fact 7.4, put aside l elements from A for reserve. Repeat the whole proof with the remaining set until Fact 7.13. Now, choose $l - \bar{l}$ arbitrary elements from the reserved set, and add their sum to the set $(\bar{l})^*A$ obtained in Fact 7.13. The resulting set is a subset of l^*A, and it contains a proper GAP as claimed in Theorem 7.1.

Now we conclude the proof of the lemma via Fact 7.13. If we assume that the first statement of the lemma does not hold, then by this fact we have that

$$\left(\frac{K}{2\times 4^d}\right)^k < f(n)\log^{d+2} n.$$

Recall that we set $K = 2^{c_1 d}$ where c_1 is a constant. By setting c_1 sufficiently large compared to $1/\nu$, it follows that

$$K^k \leq f(n)^{1+\nu}\log_2^{\alpha} n,$$

for some constant $\alpha = \alpha(\nu, d)$, proving (37). □

7.14. **Proof of Fact 7.13.** To prove Fact 7.13, let us set $l^{'} = \epsilon\min(\frac{l}{l_k}, m_k/2)$, where ϵ is a sufficiently small positive constant. Without loss of generality, we can assume that l' is an integer. The definition of l' and the construction of the Q_i's imply that for a proper choice of ϵ, $l'Q$ is a translation of a subset of $(\bar{l})^*A$ for some $\bar{l} \leq l$. Fact 7.13 follows from Theorem 3.12 and the following fact.

Fact 7.15. If $\left(\frac{K}{2\times 4^d}\right)^k \geq f(n)\log_2^{d+2} n$, then the two inequalities

$$(l')^d|Q| \gg l_k n, \tag{39}$$

$$(l')^{d'}|Q| \geq l^{d'}|A|, 1 \leq d' \leq d \tag{40}$$

hold, where $(l')^d|Q| \gg l_k n$ means that $\frac{(l')^d|Q|}{l_k n}$ tends to infinity with n.

We need to define l' as above due to the following reason. The tree might be too tall (having much more than $\log_2(l/l_0)$ levels) or too short (having less than $\log_2(l/l_0)$ levels). In the first case we have to look at some immediate level between the root and the leaves. This corresponds to the case $l' = \epsilon(l/l_k)$. In the second case, we look at some level very close to the root, and this corresponds to the definition $l' = \epsilon(m_k/2)$.

Proof of Fact 7.15. Consider an arbitrary integer d' between 1 and d. The definition of l' naturally leads to the following two cases:

Case 1. $l/l_k \le m_k/2$. In this case $l' = \epsilon(l/l_k)$. Recalling that there is a constant β such that $|Q| \ge \beta K^k b_0$ (see two paragraphs above Fact 7.13), we have that for any $d' \ge 1$

$$(l')^{d'}|Q| \ge \epsilon^{d'}(\frac{l}{l_k})^{d'} \times \beta K^k b_0 = \frac{\epsilon^{d'}\beta}{2} l^{d'}|A|\frac{K^k}{l_k^{d'}}, \tag{41}$$

where in the last equation we use the fact that $b_0 = |A|/2$. On the other hand, recall that $l_0 = 10\log_2|A|$. We have

$$l_k \le 2^{k+1}l_0 = 20 \times 2^k \log_2|A| \le 20 \times 2^k \log_2 n.$$

So, it follows from (41) that for any $1 \le d' \le d$

$$(l')^{d'}|Q| \ge \frac{\epsilon^{d'}\beta}{2 \times 20^{d'}} l^{d'}|A|\frac{K^k}{2^{kd'}\log_2^{d'} n} \ge \frac{\epsilon^{d}\beta}{2 \times 20^{d}} l^{d'}|A|\Big(\frac{K}{2^d}\Big)^k \frac{1}{\log_2^d n}, \tag{42}$$

where the second inequality follows from the assumption that $d' \le d$. The assumption on K in Fact 7.15 implies that

$$\Big(\frac{K}{2^d}\Big)^k \ge f(n)\log_2^{d+2} n > (\frac{\epsilon^d\beta}{2 \times 20^d})^{-1}\log_2^d n,$$

so the rightmost formula in (42) is larger than $l^{d'}|A|$, for any $1 \le d' \le d$. This proves the second inequality in Fact 7.15. To verify the first inequality, notice that (42) implies

$$\frac{(l')^d|Q|}{l_k} \ge \frac{\epsilon^d\beta}{2 \times 20^d} l^d|A|\Big(\frac{K}{2^d}\Big)^k \frac{1}{l_k\log^d n} \ge \frac{\epsilon^d\beta}{2 \times 20^d}\Big(\frac{K}{2^d}\Big)^k \frac{l^d|A|}{l_k\log^d n}. \tag{43}$$

Since $l_k \le 20 \times 2^k \log_2 n$, it follows that

$$\frac{(l')^d|Q|}{l_k} \ge \frac{\epsilon^d\beta}{40 \times 20^d}\Big(\frac{K}{2^{d+1}}\Big)^k \frac{l^d|A|}{\log^{d+1} n}. \tag{44}$$

The assumption on K implies that $\Big(\frac{K}{2^{d+1}}\Big)^k \ge f(n)\log^{d+2} n$, so the rightmost formula in (42) is at least

$$\frac{\epsilon^d\beta}{40 \times 20^d}\frac{l^d|A|f(n)\log^{d+2} n}{\log^{d+1} n} \gg n,$$

due to the assumption $l^d|A|f(n) \ge n$ of Lemma 7.6. This verifies the first inequality and completes the treatment of Case 1.

Case 2. $l/l_k > m_k/2$. In this case $l' = \epsilon(m_k/2)$. Since $m_k = m_0/4^k$ and

$$m_0 \ge |A|/6l_0 = |A|/60\log_2 n,$$

we have that

$$l' \ge \frac{\epsilon m_0}{2 \times 4^k} = \frac{\epsilon|A|}{120 \times 4^k \log_2 n}.$$

So for any $1 \leq d' \leq d$

$$(l')^{d'}|Q| \geq \Big(\frac{\epsilon|A|}{4^k \times 120 \log n}\Big)^{d'} \beta K^k \frac{|A|}{2} \quad = \quad \frac{\epsilon^{d'}\beta}{2 \times 120^{d'}}|A|^{d'+1}(\frac{K}{4^{d'}})^k \frac{1}{\log_2^{d'} n}$$
$$\geq \quad \frac{\epsilon^{d'}\beta}{2 \times 120^{d'}} l^{d'}|A|(\frac{K}{4^d})^k \frac{1}{\log_2^d n}.$$

Similar to the previous case, the assumption on K guarantees that $(\frac{K}{4^d})^k \geq \log_2^{d+2} n \gg \log_2^d n$, which implies that

$$\frac{\epsilon^{d'}\beta}{4 \times 120^{d'}} l^{d'}|A|(\frac{K}{4^d})^k \frac{1}{\log_2^d n} \gg l^{d'}|A|,$$

for any $1 \leq d' \leq d$, which proves the second inequality in Fact 7.15. To verify the first inequality, notice that

$$\frac{(l')^d|Q|}{l_k} \geq \frac{\epsilon^d\beta}{2 \times 120^d} l^d|A|(\frac{K}{4^d})^k \frac{1}{l_k \log_2^d n}. \tag{45}$$

Similar to the pervious case, we use the estimate $l_k \leq 20 \times 2^k \log_2 n$. This and (45) give

$$\frac{(l')^d|Q|}{l_k} \geq \frac{\epsilon^d\beta}{40 \times 120^d} l^d|A|(\frac{K}{2 \times 4^d})^k \frac{1}{\log^{d+1} n}.$$

Here we need the full strength of the assumption on K: $(\frac{K}{2\times 4^d})^k \geq \log_2^{d+2} n$. From this and the assumption that $l^d|A|f(n) \geq n$, it follows that

$$\frac{(l')^d|Q|}{l_k} \geq \frac{\epsilon^d\beta}{40 \times 120^d} \frac{l^d|A|f(n)\log_2^{d+2} n}{\log_2^{d+1} n} \geq \frac{\epsilon^d\beta}{40 \times 120^d} n \log_2 n \gg n,$$

completing the proof. □

8. Proof of Theorem 7.1 (continued)

Thanks to Corollary 7.7, from now on we can assume that A is a subset of a GAP P of rank d_1 and volume at most $|A|\log^\alpha n$, where both d_1 and α are constants depending on d. We first use this structural property to create a set B whose elements have high multiplicity with respect to A. The set B is a candidate for the set A' in the perfect triplet that we desire. After having created B, the remaining (and also the hard) part of the proof is to show that there is a sufficiently large $l' \leq l/2$ such that each elements of $l'B$ can be represented as a sum of $2l'$ distinct elements of A. This part requires a non-trivial extension of the tiling argument used in our earlier paper [28]. In order to carry out this extension, we need to prove some new properties of proper GAPs.

This section is organized as follows. In subsection 8.1, we define the set B and derive several properties of this set. This subsection also contains a proof of the theorem for the case when l is relatively small compared to $|A|$ (see Corollary 8.2). Subsection 8.3 is devoted to the study of proper GAPs. The results of this subsection will be used in the next subsection (8.6) to prove further properties of the set B. In subsection 8.7, we specify a plan for constructing a sumset $l'B$ as desired. This plan is executed in the next three subsections, 8.8, 8.10, and 8.11. The final subsection, subsection 8.12, discusses a common generalization of Theorem 5.1 and Theorem 7.1.

8.1. **Sets with high multiplicity.** We are going to show that there is a large set of which every element has high multiplicity with respect to A. Consider a monotone sequence $m_1, m_2, \ldots,$ and let S_i be the set of numbers with multiplicities between m_i and m_{i+1}. A natural way to find a large set with high multiplicity is to set $m_i = 2^i$ and process as in subsection 7.3. Here, however, we shall set the m_i's somewhat differently, in order to serve a purpose which will become clear later.

We define $m_i = \frac{|A|}{2^i i}$ for all $i = 1, 2, \ldots, \log_2 |A|$ (observe that the sequence m_i is decreasing). Let S_i be the set of those numbers whose multiplicities with respect to A are less than m_i and at least m_{i+1}. A simple double counting shows

$$\sum_{i=1}^{\log_2 |A|} m_i |S_i| \geq \binom{|A|}{2}. \tag{46}$$

Now we are going to make some use of the structure of A. Since A is a subset of a GAP P, $2A$ is a subset of $2P$. On the other hand, as P is a GAP of constant rank and volume $O(|A| \log_2^{\alpha} n)$, so $2P$ is a GAP with the same rank and volume $O(|A| \log_2^{\alpha} n)$. The set S_i (for all i) is a subset of $2A$, so it follows that

$$|S_i| \leq |2A| \leq |2P| = O(\mathrm{Vol}(2P)) = O(|A| \log_2^{\alpha} n). \tag{47}$$

By (47), the sum of those $m_i|S_i|$ where $m_i \leq \frac{|A|}{\log_2^{\alpha+2} n}$ is at most

$$O\Big(\frac{|A|}{\log_2^{\alpha+2} n} |A| \log_2^{\alpha} n\Big) \times \log |A| = O\Big(\frac{|A|}{\log_2^{\alpha+2} n} |A| \log_2^{\alpha+1} n\Big) = o(|A|^2). \tag{48}$$

This estimate allows us to omit these terms from the sum in (46) and so significantly reduce the number of terms in the sum. Notice that for any $i > \log_2 \log_2^{\alpha+2} n$, $m_i \leq \frac{|A|}{\log_2^{\alpha+2} n}$, so we only have to look at the small i's, $i \leq \log_2 \log_2^{\alpha+2} n$. From (46) and (48), we have

$$\sum_{i=1}^{\log_2 \log_2^{\alpha+2} n} \frac{|A|}{i2^i} |S_i| = \sum_{i=1}^{\log_2 \log_2^{\alpha+2} n} m_i |S_i| \geq \binom{|A|}{2} - o(|A|^2) = (\frac{1}{2} - o(1))|A|^2. \tag{49}$$

The fact that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \pi^2/6$ and (49) imply that there should be an index $1 \leq i \leq \log \log^{\alpha+2} n$ so that

$$|S_i| \geq \frac{6}{\pi^2} \frac{2^i}{i} (\frac{1}{2} - o(1))|A| > \frac{2^i}{4i}|A|.$$

Choose the smallest i satisfying the above inequality, and rename the corresponding set S_i to B. We are going to work with B in the rest of the proof. We set $l_1 = \frac{|A|}{(i+1)2^{i+1}}$. Since we shall use the letter i as an index later, let us set $t = 2^{i+1}$ to avoid confusion. Under this new notation, $l_1 = \frac{|A|}{t \log_2 t}$, where $t = 2^{i+1}$ is at most $2^{\log_2 \log_2^{\alpha+2} |A|} \leq \log_2^{\alpha+2} n$. By the definition of the S_i's, every element of B has multiplicity at least l_1 with respect to A. This implies that kB is a subset of $(2k)^*A$ for any $k \leq l_1$. Now let us consider two cases:

Case 1: $l \leq 2l_1$. In this case, we set $A' = B$, $l' = l/2$, and $n' = 2n$, and we follow the plan described in subsection 7.2. It is easy to verify that the triplet (A', l', n') is perfect. Thus we have the following corollary, which proves Theorem 7.1 for the case when l is relatively small compared to $|A|$.

Corollary 8.2. *For any fixed positive integer d there are positive constants C, c, and β depending on d such that the following holds. For any positive integers n and l and any set $A \subset [n]$ satisfying $l \le \frac{|A|}{\log^{\beta} n}$ and $l^d|A| \ge Cn$, l^*A contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some $1 \le d' \le d$.*

In the remaining part of the paper, we consider the case $l \ge 2l_1$. Before going to the next subsection, let us summarize what we have at this stage. We have created a set $B \subset 2A \subset [2n]$ where

- B has at least $\frac{|A|t}{4\log_2 t}$ elements,
- each element of B has multiplicity at least $l_1 = \frac{|A|}{t\log_2 t}$ with respect to A,
- $t \le \log_2^{\alpha+2} n$.

8.3. **Proper GAPs revisited.** If A and $2A$ are subsets of a normal GAP Q, it is tempting to conclude that A is a subset of $\frac{1}{2}Q$. A naive "proof" would go as follows: Assume that there is an element $x \in A\backslash\frac{1}{2}Q$. Since $A \subset Q$, $x \in Q\backslash\frac{1}{2}Q$, and so $2x \in 2Q\backslash Q$. But $2x \in 2A \subset Q$, a contradiction.

The trap is in the second sentence. As reasonable as it sounds, the statement "$x \in Q\backslash\frac{1}{2}Q$ implies $2x \in 2Q\backslash Q$" is not true. It is not hard to work out an example where $2x \in Q \cap 2Q$. We can, however, easily avoid this subtlety. If we assume that $2Q$ is proper, then $x \in Q\backslash\frac{1}{2}Q$ indeed implies that $2x \in 2Q\backslash Q$. Thus we can conclude

Fact 8.4. If A and $2A$ are subsets of a normal GAP Q and $2Q$ is proper, then A is a subset of $\frac{1}{2}Q$.

The above fact motivates the following lemma, which is the main result of this subsection. We assume Q is normal and its edges are divisible by l, so $\frac{1}{l}Q$ can be defined.

Lemma 8.5. *For any constants d and g there are constants γ and k such that the following holds. Let B be a finite set of integers, l a positive integer, and Q a (normal) proper GAP of rank d satisfying the following statements.*

- *The union of g translations of Q covers lB.*
- *kQ is proper.*

Then there is a translation B_1 of B such that $B_1 \cap \frac{1}{l}Q$ has at least $\gamma|B|$ elements.

Proof of Lemma 8.5. We can assume, without loss of generality, that B contains 0. The normal GAP Q can be represented as $Q = \{\sum_{i=1}^{d} x_i a_i | 0 \le x_i \le n_i\}$. If lB is covered by g translations of Q, then $lB - lB$ is covered by $g_1 = g^2$ translations of $P = Q - Q$, which has the form $P = \{\sum_{i=1}^{d} x_i a_1 | -n_i \le x_i \le n_i\}$. Let $P_1 = \frac{1}{2}P$ and $P_2 = \frac{1}{2}P_1$; it is clear that P_1 is a translation of Q. Since g_1 translations of P cover $lB - lB$ and each translation of P is the union of h^d translations of P_1, $lB - lB$ is covered by $2^d g_1$ translations of P_1. Furthermore, as each translation of P_1 is the union of 2^d translations of P_2, $lB - lB$ is covered by $4^d g_1$ translations of P_2.

Since $0 \in B$, $lB - lB$ contains B. By the pigeon-hole principle, there is a translation of P_2 containing at least a $\frac{1}{4^d g_1}$ fraction of B. Equivalently, P_2 contains a set $B' \subset a + B$ where $|B'| \ge \gamma|B|$ and a is an integer. Setting $k = 2^{d+2}g_1$ and $h = 2^{d+1}g_1 + 1$, we are going to show that $B' - B'$ is a subset of $\frac{h}{l}P_1$. Since $B' - B'$

contains a subset of constant density of a translation of B and P_1 is a translation of Q, it follows that there is a translation of B which intersects $\frac{h}{l}Q$ in $\Omega(|B|)$ elements. This implies the claim of the lemma since $\frac{h}{l}Q$ is the union of $2^h = O(1)$ translations of $\frac{1}{l}Q$.

In the rest of the proof, let us assume, for the sake of a contradiction, that there is an element x of $B' - B'$ not belonging to $\frac{h}{l}P_1$. Since $B' - B'$ is a subset of $P_2 - P_2 = P_1$, x is an element of P_1. Let s_1 be the smallest positive integer such that $s_1 x \in 2P_1 \backslash P_1$. Since both $2P_1$ and P_1 are proper, s_1 is at most l/h.

Recall that B' is a subset of $a+B$. So, an element of B' has the form $a+b$ where $b \in B$. As $x \in B' - B'$, $x = b_1 - b_2$ for some $b_1, b_2 \in B$. We set $y = s_1 x$ and consider the sequence $y, 2y, 3y, \dots, \lfloor l/s_1 \rfloor y$. As $s_1 \le l/h$, $\lfloor l/s_1 \rfloor \ge h > 2^{d+1} g_1$. Each element of the above sequence has the form $rb_1 - rb_2$ for some $r \le l$. Since $0 \in B$, these elements belong to $lB - lB$. Let us now restrict ourself to the subsequence

$$y, 2y, \dots, (2^{d+1}g + 1 + 1)y.$$

Recall that $lB - lB$ is a subset of the union of $2^d g_1$ translations of P_1. The pigeonhole principle implies that there should be a translation, say $a' + P_1$, containing two elements iy and jy where $2 \le i - j \le 2^{d+1} g_1$. The difference $(i-j)y$ is an element of $(a' + P_1) - (a' + P_1) = P_1 - P_1 = 2P_1$. Since $i - j < 2^{d+1} g_1 = k/2$, $2(i-j)P_1$ is proper by the second assumption of the lemma. Moreover, y is an element of $2P_1 \backslash P_1$ so $(i-j)y$ is an element of $2(i-j)P_1 \backslash (i-j)P_1$. This is a contradiction because $(i-j)P_1$ contains $2P_1$ as $i - j \ge 2$. □

8.6. **Properties of B.** Let us consider the set $l_1 B$. By the lower bounds on l_1 and $|B|$ (see the last paragraph of subsection 8.1) we have

$$l_1^d |B| \ge \frac{|A|^{d+1}}{4t^{d-1} \log_2^{d+1} t}.$$

The assumptions $l^d |A| \ge Cn$ and $l \le |A|/2$ of Theorem 7.1 guarantee that $|A|^{d+1} \ge Cn$ and so

$$l_1^d |B| \ge \frac{C}{4t^{d-1} \log_2^{d+1} t} n.$$

The factor $t^{d-1} \log_2^{d+1} t$ is the main source of our troubles. If t is a constant bounded by a function of d (say e^{d^2}), then by increasing the value of C, we can assume that $\frac{C}{4t^{d-1} \log^{d+1} t}$ is sufficiently large and so Theorem 3.12 can by applied. However, t can be as large as a positive power of $\log_2 n$ and in general cannot be bounded by any function of d.

In the remaining part of the proof, we assume that t is very large compared to d (for all purposes, it is sufficient to assume, say, $t \ge e^{e^{100d}}$). We are going to find a way to play this assumption to our advantage (and through our arguments one will see the reason for the somewhat artificial definition of m_i's). In the remaining part of this subsection, we use Lemma 8.5 to derive some properties of B which are useful for us.

Let us start with the usual "doubling" trick. Set $B_0 = B$ and define $B_{i+1} = 2B_i$. We claim that at some stage we will be in a position to apply Lemma 8.5.

It is easy to show (using an argument similar to those used in the proof of Theorem 3.12) that there is some s such that $2^s \ll l_1$ satisfying

$$|2B_s| \le (2^{d+2} - 1)|B_s|.$$

As usual, we let s be the smallest number with this property. By Lemma 4.9, B_s is a subset of a constant number of translations of a GAP P_0 of rank $d+1$ where $\mathrm{Vol}(P_0) = O(|B_s|)$. Moreover, the proper filling lemma implies that there is a constant g_1 so that $g_1 B_s$ contains a proper GAP P_1 of rank $d+1$ whose volume is $\Theta(|B_s|)$. The differences of P_1 are constant multiples of the corresponding differences of P_0, so P_0 is covered by a constant number of translations of P_1. Therefore, B_s is covered by a constant number of translations of the proper GAP P_1.

In order to apply Lemma 8.5, we also need the assumption that there is a sufficiently large constant k_1 such that $k_1 P_1$ is proper. Unfortunately, nothing guarantees the existence of k_1. However, if we cannot find k_1, then we can use our "rank reduction" argument. Set k_1 be a sufficiently large constant and consider the sequence $P_1, 2P_1, 4P_1, \dots$. If for some $i \leq \log_2 k_1$, $2^i P_1$ fails to be proper, then by the rank reduction argument, we can find a proper GAP P_2 of rank strictly less than the rank of P_1 such that the following two properties hold:

- There is a constant g_2 such that $g_2 P_1$ contains P_2.
- A constant number of translations of P_2 covers P_1.

It follows that a constant number of translations of P_2 covers B_s. Now repeat the above argument with P_2. As the rank decreases each time, we should be done after a constant number of steps. According to our arguments, the final proper GAP (for which the assumptions of Lemma 8.5 are satisfied) still has volume $\Omega(|B_s|)$. We call this final GAP P'.

By applying Lemma 8.5 to P', we obtain a few new properties of B:

- For some $m = O(2^s)$, mB contains a GAP P' which has volume at least
$$\Omega(|B_s|) = \Omega\Big((2^{d+2}-1)^s |B|\Big) = \Omega(2^{s(d+1)}|B|).$$
Moreover, since $l_1 \gg 2^s$, $m \ll l_1$.
- There is a subset B' of B such that $|B'| \geq \gamma |B|$ and B' is a subset of a GAP P which is a translation of $\frac{1}{m}P'$.

Since we are allowed to ignore constant factors, we assume that $B' = B$ for convenience. Moreover, without loss of generality, we could assume that P' has symmetric form, namely, $P' = \{a_1 x_1 + \cdots + a_{d_1} x_{d_1} | -n_i \leq x_i \leq n_i\}$.

8.7. **A plan.** Let us now give a rough discussion of our plan:

- We are going to find a set $\mathcal{T}$ of l_2-tuples in B (a k-tuple is a set of k not necessarily different elements) such that the sum of the elements in any tuple is an element of $(2l_2)^* A$, where $l_2 \gg l_1$ is a parameter to be defined. Let $\mathcal{S}$ be the collection of the sums of the tuples in $\mathcal{T}$. We create $\mathcal{T}$ in a particular manner so that $\mathcal{S}$ is sufficiently dense in $l_2 B$.
- We next prove that $\mathcal{S} + l_1 B$ contains $l_2 B$, relying on the fact that $\mathcal{S}$ is dense in $l_2 B$. This way we obtain the sumset $l_2 B$ where l_2 is significantly larger than l_1.
- Since $\mathcal{S}$ is a subset of $(2l_2)^* A$ and $l_1 B$ is a subset of $(2l_1)^* A$, $\mathcal{S} + l_1 B$ is a subset of $(2l_2)^* A + (2l_1)^* A$. The obvious obstacle here is that the same element of A might be used twice, once in $(2l_2)^* A$ and once in $(2l_1)^* A$. We overcome this problem in subsection 8.10 and show that $l_2 B$ is in fact an element of $(2l_2 + 2l_1)^* A$.

We call this plan a tiling operation as what it does is to tile many copies of l_1B together to get a bigger set l_2B.

Would we be done after a successful implementation of this plan ? Well, we would be in a very good position if we can guarantee that $l_2^d|B| \gg n$ (this inequality is necessary for an application of Theorem 3.12 to l_2B). In the case $d = 1$, we can do this, and the above plan was carried out successfully in an earlier paper [27]. Unfortunately, there is a serious difference between the two cases case $d = 1$ and $d \geq 2$. For $d = 1$, the troublesome factor $t^{d-1}\log_2^{d+1} t$ is only $\log_2^2 t$, and there is a way to set up l_2 so this polylogarithmic factor can be ignored. On the other hand, in the general case $d \geq 2$, the troublesome factor is a polynomial in t (which is of a different order of magnitude) and even the optimal value we could get for l_2 would not be enough to kill this factor.

We are going to resolve this problem by repeating the second step of the plan many times. Roughly speaking, what we shall do is to put many original tiles (copies of the set l_1B) together to get a larger tile l_2B. Next, we put many copies of l_2B together to get an even larger tile l_3B and so on. We repeat the operation until we get a sufficiently large tile l_kB which satisfies $l_k^d|B| \gg n$.

There is a trade-off in this argument. The repetitions make the problem mentioned in the last step of the above plan more severe: Now the same element of A might be used as many as k times. Luckily, our treatment for this problem is not sensitive to this modification as far as k remains a constant, which is the case.

Finally, let us go back to address the first step: How can we find l_2 elements of B such that their sum can be represented as the sum of $2l_2$ different elements of A? The main idea is as follows: An element of B has multiplicity l_1 with respect to A, so it gives us l_1 pairs of elements of A, all having the same sum. Therefore, a set of m different elements of B gives us l_1m different pairs. On the other hand, each element in A occurs in at most $|A| - 1 < |A|$ pairs. Using the greedy algorithm, we can find at least $\frac{l_1m}{2|A|}$ mutually disjoint pairs. Thus, for any $l_2 \leq \frac{l_1m}{2|A|}$, we have a collection of l_2 mutually disjoint pairs. Clearly, the sum of the l_2 elements of B corresponding to these pairs is an elements of $(2l_2)^*A$.

The critical feature of this step is how to choose the set of m elements of B. We discuss this issue in the next paragraph.

8.8. **The tiling operation: Start.** Let us start with the execution of the first step. Recall, from the last paragraph of subsection 8.6, that B is a subset of a proper GAP P of constant rank d_1 (the value of d_1 is irrelevant, but we do know that $d_1 \leq d + 1$). It is easier for the reader to visualize the argument if he/she identifies P with a d_1-dimensional box. Partition each edge of P into T_1 intervals of equal length, where T_1 is a parameter to be determined. The products of these intervals partition P into $(T_1)^{d_1}$ identical small boxes. A small box Q is *dense* if the number of elements of B in Q is at least $\frac{|B|}{2(T_1)^{d_1}}$; Q is *sparse* otherwise. The sparse boxes contain at most half of the elements of B, so at least half of the elements of B should be contained in dense boxes. Since constants like 1/2 do not play any significant role, we assume, for the sake of convenience, that all elements of B are contained in dense boxes.

Let us recall that $|B| \geq \frac{|A|t}{4\log_2 t}$ and $l_1 = \frac{|A|}{t\log_2 t}$. By throwing away dummy elements, we can assume that $|B|$ is exactly $\frac{|A|t}{4\log_2 t}$.

Consider a dense box Q. For each element $x \in B \cap Q$, x has multiplicity l_1 with respect to A. We set the number m in the last paragraph of the previous subsection to be $|B|/2T_1^{d_1}$; as Q is dense we are guaranteed to find this many elements of B in Q. The argument in the above-mentioned paragraph shows that we can have at least

$$\frac{l_1|B|}{4(T_1)^{d_1}|A|}$$

disjoint pairs. For technical reasons, we do not set l_2 equal to this value but equal to one-third of it:

$$l_2 = \frac{l_1|B|}{12(T_1)^{d_1}|A|}.$$

For $x \in B$ let N_x be the collection of pairs (in A) summing up to x. We have proved

Fact 8.9. For each dense box Q, the union of N_x's ($x \in B \cap Q$) contains at least $3l_2$ mutually disjoint pairs.

Substituting the values of l_1 and $|B|$ into the formula of l_2, we have

$$l_2 = \frac{|A|}{48(T_1)^{d_1}\log_2^2 t}. \tag{50}$$

For each dense box Q, fix a collection N_Q of $3l_2$ disjoint pairs. For a pair (a, b) in N_Q, the number $a + b$ is a point of the box Q ($a + b \in B \cap Q$). In the following, we denote by D_Q the collection of these points; D_Q is a multi-set as different pairs may have the same sum. Let D be the union of the D_Q's.

Let us now take a closer look at the set l_2B. An element x of this set can be written as $x = x_1 + \cdots + x_{l_2}$, where the x_i's are not necessarily different elements of B. Moreover, we assumed that every element of B is in some dense box, so each x_i is in some dense box Q (different x_i's may, of course, belong to different boxes). Fix a dense box Q; for each $x_i \in Q$, we are going to replace it by some $y_i \in D_Q$. Now comes a very important point. Since $|D_Q| \geq 3l_2$ for any dense box Q, we can replace $x_1, \dots, x_{l_2}$ with elements $y_1, \dots, y_{l_2}$ with the following property: There are mutually disjoint pairs $(a_1, a'_1), \dots, (a_{l_2}, a'_{l_2})$, $a_i, a'_i \in A$, such that $a_i + a'_i = y_i$. To see this, let us consider the following rule. For x_1, choose an arbitrary pair (a_1, a'_1) from D_{Q_1} where Q_1 is the dense box containing x_1; set $y_1 = a_1 + a'_1$. Assume that $(a_1, a'_1), \dots, (a_{i-1}, a'_{i-1})$ have been chosen. Consider x_i and the set D_{Q_i} where Q_i is the dense box containing x_i. Delete from D_{Q_i} every pair which has a non-empty intersection with the chosen pairs. Since the pairs in D_{Q_i} are disjoint, any pair (a_j, a'_j) $(1 \leq j \leq i-1)$ could intersect at most two pairs in D_{Q_i} so we delete at most

$$2(i-1) \leq 2(l_2 - 1) < 2l_2$$

pairs from D_{Q_i}. But D_{Q_i} contains $3l_2$ pairs so there are always some pairs left and we choose an arbitrary one among these.

The disjointness of the chosen pairs guarantees that $y = y_1 + \cdots + y_{l_2}$ can be represented as a sum of exactly $2l_2$ different elements from A. Let $\mathcal{T}$ denote the collection of the tuples $(y_1, \dots, y_{l_2})$, and let $\mathcal{S}$ be the collections of their sums. Following the plan, we next show that $\mathcal{S} + l_1B$ contains l_2B.

Consider $x = x_1 + \cdots + x_{l_2}$. Since $x_i \in B$ and $B \subset P$, each x_i is an element of the box P and can be viewed as a point in $\mathbb{Z}^{d_1}$, so we can view x as a vector in $\mathbb{Z}^{d_1}$. By replacing x_i with y_i, we obtain another vector $y = \sum_{i=1}^{l_2} y_i$. We are going to find a box P_1 centered at the origin so that P_1 is a subset of $l_1 B$ and the difference $x - y = \sum_{i=1}^{l_2}(x_i - y_i)$ is a vector in P_1. The union of the copies of such a P_1 centered at the points of $\mathcal{S}$ cover $l_2 B$. As $P_1 \subset l_1 B$, it follows that $l_2 B \subset \mathcal{S} + l_1 B$, as desired.

The key observation in what follows is that $x_i - y_i$ is small because they are in the same small box (this is the main reason for why we partition P into many small boxes). Let us fix an edge of P and assume that its length is s_1. The absolute value of the component of $x_i - y_i$ in the direction of this edge is at most s_1/T_1. It follows that the corresponding component of $x - y$ is at most $l_2 s_1/T_1$. We are going to choose T_1 and define P_1 so that this bound is at most half the length of the corresponding edge of P_1 (P_1 is centered at the origin). This would imply that P_1 contains the vector $x - y$.

Now we are going to define P_1. The last paragraph of subsection 8.6 tells us that mB contains a GAP $P' = mP$, for some $m \ll l_1$. Thus $l_1 B$ contains the box $\frac{l_1}{m}P' = l_1 P$. This is our box P_1. Observe that P_1's edge in the relevant direction has length $s_1 l_1$. In order to guarantee that this length is at least twice $l_2 s_1/T_1$, we should set T_1 so that

$$s_1 l_1 \geq \frac{2 l_2 s_1}{T_1} = \frac{s_1 |A|}{24 T_1^{d_1+1} \log_2^2 t}. \tag{51}$$

To satisfy (51), it is sufficient to set

$$T_1 = \Big(\frac{|A|}{24 l_1 \log_2^2 t}\Big)^{1/(d_1+1)} = \Big(\frac{t}{24 \log_2 t}\Big)^{1/(d_1+1)},$$

since $l_1 = |A|/t \log_1 t$. For the sake of a cleaner calculation, we set T_1 a little bit larger:

$$T_1 = t^{1/(d_1+1)}.$$

Substituting the above value of T_1 into the definition of l_2 in (50), we obtain

$$l_2 = \frac{|A|}{48 (T_1)^{d_1} \log_2^2 t} = \frac{|A|}{48 t^{d_1/(d_1+1)} \log_2^2 t} \geq \frac{|A|}{t^{d_1/(d_1+1)} \log_2^2 t}. \tag{52}$$

This l_2 is still not large enough, namely, $l_2^d |B|$ could still be smaller than n. Indeed, the above lower bound on l_2 only guarantees that

$$l_2^d |B| \geq \frac{|A|^d}{t^{dd_1/(d_1+1)} \log^{2d} t} \times \frac{|A| t}{8 \log_2 t} = \Theta(\frac{|A|^{d+1}}{t^{dd_1/(d_1+1)-1} \log_2^{2d+1} t}), \tag{53}$$

where the right-hand side can be significantly smaller than n if $|A| = O(n^{1/(d+1)})$ and $dd_1/(d_1+1) - 1 > 0$. Our plan is to increase the value of l_2 by repeated tiling.

To conclude this subsection, let us discuss the problem that the same element of A might appear twice in a representation of an element of $l_2 B$. Observe that $l_2 B$ is a subset of $(2l_1 + 2l_2)^* A$, and thus any element of $l_2 B$ is a sum of $2l_1 + 2l_2$ elements of A. However, as we already pointed out, an element of A can appear twice, once in $(2l_1)^* A$ and once in $(2l_1)^* A$. This problem can be resolved by the so-called *cloning* trick, introduced in [28].

8.10. **The cloning argument.** At the very beginning of the entire proof, we split the set A into two sets A' and $A^{''}$ in such a way that $|A'| \approx |A^{''}|$ and any number x which has high multiplicity with respect to A' should have almost the same multiplicity with respect to $A^{''}$. Next, we continue with A' and keep $A^{''}$ for reserve.

Repeat the entire proof with A' playing the role of A until the last paragraph above subsection 8.10. We call the set of elements with high multiplicity (with respect to A') B' instead of B. Now doing the same with $A^{''}$, we obtain a set $B^{''}$.

The key point now is that with a proper splitting, the two sets B' and $B^{''}$ are exactly the same. So when we look at l_2B' as a subset of $\mathcal{S}+l_1B'$, we can think of an element of $\mathcal{S}$ as a sum of l_2 elements from $B^{''}$, rather than from B'. Therefore, when we replace each element from B' and $B^{''}$ by the sum of two elements from A, the elements used for $\mathcal{S}$ come from $A^{''}$ and the elements used for l_1B' come from A', and this guarantees that no element of A is used twice.

A random splitting provides the sets A' and $A^{''}$ as required. For each element of A throw a fair coin. If heads, we put it into A'; otherwise it goes to $A^{''}$. If a number x has multiplicity $m_x \gg \log n$ with respect to A, then standard large deviation inequalities (such as Chernoff's) tell us that with probability at least $1-n^{-2}$, x has multiplicities $\frac{m_x}{4} \pm 10\sqrt{m_x \log n} = (1+o(1))\frac{m_x}{4}$ with respect to both A' and $A^{''}$. Since there are only at most $2n$ numbers x to consider, with probability close to 1, every x with multiplicity $\gg \log n$ has approximately the same multiplicities in A' and $A^{''}$.

When we create the set S_i (which we later rename to B) in subsection 8.1, any element x in S_i has multiplicity $m(x)$ at least $\frac{|A|}{2^{i+1}(i+1)} \gg \log n$ with respect to A. So x will have multiplicity roughly $m(x)/4$ with respect to both A' and $A^{''}$. Thus one can expect that x will appear in both B' and $B^{''}$. The only case we may have to worry about is when $m(x)$ is very close to a threshold (say m_i) and then (because the error terms can go either way) x might be in B' but not in $B^{''}$ (or vice versa). This problem is easy to deal with. We just force this x to be in both B' and $B^{''}$ (of course, forcing x might decrease l_1 slightly (by a factor .9, say) but this does not influence anything).

8.11. **The tiling operation: Finish.** We repeat the tiling operation in subsection 8.8 with new parameters. Now P is cut into $T_2^{d_1}$ boxes, where T_2 is a parameter to be chosen. Instead of (50), we define

$$l_3 = \frac{|A|}{48(T_2)^{d_1}\log_2^2 t}. \tag{54}$$

Here is our key point: in order to obtain l_3B, we now add $\mathcal{S}$ with l_2B, instead of with l_1B as in subsection 8.8. This means that instead of P_1 we can use the larger box $P_2 = \frac{l_2}{l_1}P_1$. As an analogue of (51), the condition we need on T_2 is

$$s_1 l_2 \geq \frac{2l_3 s_1}{T_2}. \tag{55}$$

Notice that in the left-hand side of (55) we have l_2 instead of l_1. The fact that $l_2 \gg l_1$ allows us to set T_2 much smaller than T_1. Consequently, l_3 becomes significantly larger than l_2. Repeating this results in a sequence $l_1 < l_2 < l_3 < l_4 < \dots$, where for some constant k, l_k will be sufficiently large.

Now let us present some computation. The derivation of T_2 from (55) is similar to that of T_1 from (51). It is sufficient to set

$$T_2 = \Big(\frac{|A|}{24 l_2 \log_2^2 t}\Big)^{1/(d_1+1)}$$

in order to satisfy (55). Since $l_2 \geq \frac{|A|}{t^{d_1/d_1+1}\log_2^2 t}$,

$$\Big(\frac{|A|}{24 l_2 \log_2^2 t}\Big)^{1/(d_1+1)} \leq \Big(\frac{t^{d_1/(d_1+1)}}{24}\Big)^{1/(d_1+1)},$$

so we can set $T_2 = \Big(\frac{t^{d_1/(d_1+1)}t}{24}\Big)^{1/(d_1+1)}$. Again, for convenience, we set T_2 a bit larger:

$$T_2 = t^{d_1/(d_1+1)^2},$$

which implies

$$l_3 = \frac{|A|}{48(T_2)^{d_1}\log_2^2 t} \geq \frac{|A|}{t^{d_1^2/(d_1+1)^2}\log_2^2 t}. \tag{56}$$

By induction, we can show that

$$l_k \geq \frac{|A|}{t^{d_1^k/(d_1+1)^k}\log_2^2 t}. \tag{57}$$

By choosing k sufficiently large (say, $k = 2(d_1+1)\log(d+1)$), we have (using the fact that t is much larger than d)

$$l_k \geq \frac{|A|}{t^{1/2(d+1)}\log_2^2 t} \geq \frac{|A|}{t^{1/2d}}.$$

Now l_k is sufficiently large, namely, it satisfies the critical inequality $l_k^d|B| \gg n$ (one can easily check this by substituting $|B| = \frac{|A|t}{4\log t}$). This inequality provides the necessary condition we need to apply Theorem 3.12 to the set l_kB.

Our proof shows that l_kB is a subset of $(2l_k)^*A + (2l_{k-1})^*A + \cdots + (2l_1)^*A$. In this sum an element of A might be used k times. This problem can be handled using the cloning argument exactly as before, with the only formal modification that instead of splitting A into two subsets, we split it into k subsets.

To be completely finished, there is one last issue we need to discuss and that is the magnitude of the sum $l_1 + \cdots + l_k$.

As we have shown (with the aid of cloning), the set l_kB is a subset of

$$(2l_1 + \cdots + 2l_k)^*A = \tilde{l}^*A,$$

where $\tilde{l} = 2l_1 + \cdots + 2l_k$. We need to compare $\tilde{l}$ with l, and naturally there are two cases. If $\tilde{l} \leq l$, then we set $A' = B$, $l' = \tilde{l}$, and $n' = 2n$. In this case, we have

$$\begin{aligned}
(l')^{d'}|A'| &\geq l_k^{d'}|B| \\
&\geq (\frac{|A|}{t^{1/2d}})^{d'}|B| \\
&\geq (\frac{|A|}{t^{1/2d}})^{d'} \times \frac{|A|t}{4\log_2 t} \\
&\geq |A|^{d'+1}t^{1-(d'+1)/2d} \text{ (as } t \text{ is much larger than } \log_2 t) \\
&\geq |A|^{d'+1} \\
&\geq l^{d'}|A|,
\end{aligned}$$

for every $1 \leq d' \leq d$. This guarantees that the triple (A', l', n') is perfect.

In the remaining case when $\tilde{l} > l$, there is an index $i < k$ such that

$$2l_1 + \cdots + 2l_i \leq l < 2l_1 + \cdots + 2l_{i+1}.$$

We now modify the tiling operation a little bit. First of all, it is clear that we do not have to proceed beyond the ith tiling, so we make this tiling our last. Moreover, in this last tiling we shall not use the whole set l_iB as a tile, but only a fraction of it, say l'_iB for some $l'_i < l_i$ (as we mentioned many times, our arguments are invariant with respect to translations, so we can assume that l'_iB is a subset of l_iB). As the result, obtain a set $l'_{i+1}B$ instead of $l_{i+1}B$, for some $l'_{i+1} \leq l_{i+1}$. The set $l'_{i+1}B$ is a subset of $(2l_1 + \cdots + 2l_i + 2l'_{i+1})^*A$ where, with a proper choice of l'_i, we can guarantee that

$$l/2 \leq (2l_1 + \cdots + 2l_i + 2l'_{i+1}) \leq l.$$

Now we can set $A' = B$, $l' = l'_{i+1}$, $n' = 2n$ and conclude the proof as discussed in subsection 7.2. □

8.12. **A common generalization of Theorems 5.1 and 7.1.** In this subsection, we present a common generalization of Theorems 5.1 and 7.1. Let us first remind the reader of the sumsets studied in these two theorems. In Theorem 5.1, we consider a sum of different sets $A_1, \dots, A_l$, but allow the same number to appear many times in a representation (the same number may occur in several A_i's). On the other hand, in Theorem 7.1 we have only one set A in the sum, but with the restriction that the summands of a representation must be different. For a common generalization of these theorems, we consider a sum which involves different elements of different sets. Let $A_1, \dots, A_l$ be sets of integers. We define $A_1 \overset{*}{+} A_2 \overset{*}{+} \cdots \overset{*}{+} A_l$ as the collection of all numbers which can be represented as a sum of l different numbers $a_1 \in A_1, \dots, a_l \in A_l$. Formally speaking,

$$A_1 \overset{*}{+} A_2 \overset{*}{+} \cdots \overset{*}{+} A_l = \{a_1 + \cdots + a_l | a_i \in A_i, a_i \neq a_j \text{ for } 1 \leq i < j \leq l\}.$$

We refer to $A_1 \overset{*}{+} A_2$ as the star sum of A_1 and A_2.

Theorem 8.13. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let $A_1, \dots, A_l$ be subsets of size $|A|$ of $[n]$ where l and $|A|$ satisfy $l^d|A| \geq Cn$. Then $A_1 \overset{*}{+} A_2 \overset{*}{+} \cdots \overset{*}{+} A_l$ contains a GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer $1 \leq d' \leq d$.*

About the proof, one's first impression would be that one can prove Theorem 8.13 using Theorem 7.1 the same way that one proved Theorem 5.1 using Theorem 3.12. This, however, is not possible due to a subtle problem involving star sums. While it is clear that the (set) equality

$$(A_1 + A_2) + (A_3 + A_4) = A_1 + A_2 + A_3 + A_4$$

is true, its star sum counterpart

$$(A_1 \overset{*}{+} A_2) \overset{*}{+} (A_3 \overset{*}{+} A_4) = A_1 \overset{*}{+} A_2 \overset{*}{+} A_3 \overset{*}{+} A_4$$

is false.

So far, the only way (we know of) to verify Theorem 8.13 is to repeat the proof of Theorem 7.1 with appropriate modifications. This is a tedious task, but no essential new arguments are required, and we thus omit the details. Let us, however, present the variant of a step in the proof of Theorem 7.1, Lemma 7.9, in order to give the reader an idea about the kind of modifications one needs to carry out.

Lemma 8.14. *Let A_i, $1 \leq i \leq 20 \log_2 |A|$, be finite sets of real numbers with the same cardinality $|A|$, where $|A|$ is sufficiently large. Then there is an integer $1 \leq T \leq 10 \log_2 |A|$ and elements $a_1 \in A_1, a_2 \in A_2, \dots, a_{2T} \in A_{2T}$ such that all a_i's are different and the set $B = \{a_1, \dots, a_{2T}\}$ satisfies*

$$|T^* B| \geq |A|.$$

Proof of Lemma 8.14. We assume $|A|$ is sufficiently large so that $|A| \geq 100 \log_2 |A|$. We choose a_1 and a_2 from A_1 and A_2, respectively, with the only condition that $a_1 \neq a_2$. Once $a_1, \dots, a_{2i}$ have been chosen, we next choose a_{2i+1} and a_{2i+2} from $A_{2i+1} \backslash \{a_1, \dots, a_{2i}\}$ and $A_{2i+2} \backslash \{a_1, \dots, a_{2i}\}$ so that $a_{2i+1} \neq a_{2i+2}$ and

$$|(i+1)^* \{a_1, \dots, a_{2i+1}, a_{2i+2}\}| \geq 1.1 |i^* \{a_1, \dots, a_{2i}\}| \tag{58}$$

(if there are many possible pairs, we choose an arbitrary one). We stop at time T when $|T^* \{a_1, \dots, a_{2T}\}| \geq |A|$ and set $B = \{a_1, \dots, a_{2T}\}$. It is clear that $|B| \leq 2 \log_{1.1} |A| \leq 20 \log_2 |A|$. The only point we need to make is to show that as long as $|i^* \{a_1, \dots, a_{2i}\}| < |A|$, we can always find a pair (a_{2i+1}, a_{2i+2}) to satisfy (58). Assume (for a contradiction) that we get stuck at the ith step, and denote by S the sum set $i^* \{a_1, \dots, a_{2i}\}$. For any two numbers $a \in A_{2i+1} \backslash \{a_1, \dots, a_{2i}\}$, $a' \in A_{2i+2} \backslash \{a_1, \dots, a_{2i}\}$ the union $(a+S) \cup (a'+S)$ is a subset of $(i+1)^* \{a_1, \dots, a_{2i}, a, a'\}$. So by the assumption we have

$$|(a+S) \cup (a'+S)| \leq 1.1 |S|.$$

Since both $a + S$ and $a' + S$ have $|S|$ elements, it follows that their intersection has at least $.9|S|$ elements. This implies that the equation $a' - a = x - y$ has at least $.9|S|$ solutions (x, y) where $x \in S$ and $y \in S$. Now let us fix a as the smallest element of $A_{2i+1} \backslash \{a_1, \dots, a_{2i}\}$ and choose a' arbitrarily from $A_{2i+2} \backslash \{a_1, \dots, a_{2i}, a\}$ (we exclude a' from A_{2i+2}, so we are guaranteed that $a \neq a'$). There are at least $|A| - 2i - 1 \geq .9|A|$ choices for a', each of which generates at least $.9|S|$ pairs (x, y) where both x and y are elements of S. As all (x, y) pairs are different, we have that

$$.9|A| \times .9|S| \leq \binom{|S|}{2},$$

which implies that $|S| > |A|$, a contradiction. This concludes the proof. □

9. Erdös's conjecture on complete sequences

In 1962, Erdös introduced the following notion, which has since become quite popular: An infinite set A of positive integers is *complete* if every sufficiently large positive integer can be represented as a sum of different elements of A (see Section 6 of [9] or Section 4.3 of [23] for surveys about completeness). For instance, Vinogradov's result (mentioned in the Overview (Section 1)) implies that the set of primes is complete. On the other hand, there is a big difference between the study of complete sequences and the study of classical problems of Vinogradov-Waring type. For completeness, we do not require the number of summands in a representation to be the same. This relaxation leads to a quite different kind of results. For problems of Vinogradov-Waring type (where the number of summands is fixed), one usually requires a very precise description of the sequence (the set of primes or the set of squares, say). For problems concerning complete sequences, it has turned out that there is much more flexibility.

What would be the first condition for a sequence to be complete ? Well, density must be the answer, as one cannot hope to represent every positive integer with a very sparse sequence. But one would also notice instantly that density itself would not be enough: The set of even numbers has very high density, but it is clearly not complete. This shows that one should also consider a condition involving modularity.

In number theory it happens quite frequently that the obvious necessary conditions are also sufficient. In 1962, Erdös made the following conjecture.

Conjecture 9.1. *There is a constant c such that the following holds. Any increasing sequence $A = \{a_1 < a_2 < a_3 < \dots\}$ satisfying*
(a) $A(n) \geq cn^{1/2}$,
(b) S_A *contains an element of every infinite arithmetic progression*
is complete.

Here and later $A(n)$ denotes the number of elements of A not exceeding n. The bound on $A(n)$ is best possible, up to the constant factor c, as shown by Cassels [4].

Erdös [8] proved that the statement of the conjecture holds if one replaces (a) by the stronger condition that $A(n) \geq cn^{(\sqrt{5}-1)/2}$. An important step was later made by Folkman [14], who improved Erdös's result by showing that $A(n) \geq cn^{1/2+\epsilon}$ is sufficient, for any positive constant ϵ. The first and simpler part in Folkman's proof is to remove condition (b). He showed that any sequence satisfying (b) could be partitioned into two subsequences with the same density, one of which still satisfies (b). In the next and critical step, Folkman shows that if A is a sequence with density at least $n^{1/2+\epsilon}$, then S_A contains an infinite arithmetic progression (in other words, A is subcomplete). His result follows immediately from these two steps. Folkman's proof, naturally, led him to the following conjecture, which is perhaps even more to the point than Conjecture 9.1

Conjecture 9.2. *There is a constant c such that the following holds. Any increasing sequence $A = \{a_1 < a_2 < a_3 < \dots\}$ satisfying $A(n) \geq cn^{1/2}$ is subcomplete.*

Folkman's result has been further strengthened recently by Hegyvári [18] and by Łuczak and Schoen [21], who (independently) reduced the density $n^{1/2+\epsilon}$ to $cn^{1/2} \log^{1/2} n$, using the result of Sárközy (see Section 3).

In a previous paper [28], we proved Conjecture 9.2. However, we discuss this problem here for pedagogical reasons. It would be more useful for the reader to consider this problem together with Conjecture 6.1 and under the general sufficient condition proved in Section 6. As a matter of fact, given this sufficient condition, it is now very simple to prove Conjecture 9.2. The only modification one needs to make is to replace Lemma 6.10 by the following.

Lemma 9.3. *There is a constant C such that the following holds. If A is a set of different positive integers between 1 and n and $|A| \geq C\sqrt{n}$, then S_A contains an arithmetic progression of length n.*

The rest of the proof is the same.

Theorem 9.4. *There is a constant c such that the following holds. Any increasing sequence $A = \{a_1 < a_2 < a_3 < \dots\}$ satisfying $A(n) \geq cn^{1/2}$ is subcomplete.*

Let us conclude with a comment on Conjecture 9.2 and Conjecture 6.1. These conjectures look quite similar, which comes as no surprise as they appeared in the same paper. The interesting point here is that the proof of Conjecture 6.1 requires only Theorem 5.1, which is an easy application of Theorem 3.12, but the proof of Conjecture 9.2 requires the much harder Theorem 7.1. On the other hand, prior to our study, Conjecture 6.1 seemed harder to attack and fewer partial results were known.

Remark. We have recently been informed by Lev (private communication) that Chen [6] also proved a variant of Theorem 9.4, using a different method.

10. Arithmetic progressions in finite fields

In this section we assume that n is a prime. We are going to extends our previous theorems to arithmetic progressions modulo n. The quantitative statements in these theorems will change slightly, but the proofs remain essentially the same. We first establish the results and then describe an application.

10.1. **Results.** In order to show why we need a modification in the statements of the theorems, let us consider the proof of Theorem 3.12. At one point in the proof (see the paragraph following (17)), we used the fact that lA is a subset of the interval $[ln]$ and thus has cardinality at most ln. In the finite field case, lA is always a subset of the set of residues modulo n, and so its cardinality is always at most n, no matter how large l is. This suggests that we should gain an extra factor l in the assumption of the theorem, and that has indeed turned out to be the case. The analogue of Theorem 3.12 is as follows.

Theorem 10.2. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let n be a prime, let l be a positive integer, and let A be a set of residues modulo n such that $l^{d+1}|A| \geq Cn$. Then the sumset lA (modulo n) contains an arithmetic progression (modulo n) of length $\min\{n, cl|A|^{1/d}\}$.*

There are two modifications in Theorem 10.2 (compared with Theorem 3.12). First we changed l^d to l^{d+1}, which is consistent with the above discussion. Second, we changed the lower bound from $cl|A|^{1/d}$ to $\min\{n, cl|A|^{1/d}\}$. This modification is natural and justified, as lA can have at most n elements. We shall comment on this at the end of the next paragraph.

The proof of Theorem 10.2 is the same as the proof of Theorem 3.12. The only place where one needs a (formal) modification is (17). In this inequality, the rightmost formula should be Cn instead of Cln, which is consistent with the discussion in the paragraph preceding Theorem 10.2. Freiman's theorem and all lemmas used for the proof of Theorem 3.12 hold for residue classes (see [27] for exact statements). To explain the change in the lower bound, notice that in the proof of Theorem 3.12 we actually showed that either $lA = [ln]$ or lA contains an arithmetic progression of length $cl|A|^{1/d}$. Its finite field analogue says that either lA contains all residues modulo n or it contains an arithmetic progression of length $cl|A|^{1/d}$. In Theorem 3.12, it is unnecessary to state the lower bound as $\min\{ln, cl|A|^{1/d}\}$ because ln is always larger than $cl|A|^{1/d}$. On the other hand, in the finite field case, it makes sense to write $\min\{n, cl|A|^{1/d}\}$ since n can be smaller than $cl|A|^{1/d}$.

Theorem 10.2 demonstrates the flexibility of our method. It is not clear, for instance, how to prove a finite field version of Theorem 3.3 (which is a special case of Theorem 3.12) using the original approaches of Freiman and Sárközy.

Similar to Theorem 3.12, Theorem 10.2 is sharp. One can modify the general construction in Section 3 to match the lower bound. This construction also mirrors the extra term l.

A construction modulo n. We present a modification of the principal construction in Section 3. Now set $a = \lfloor \frac{(1-\delta/3)n/l}{d|A|^{1/d}} \rfloor$ (notice the extra l in the nominator) and $b = \lfloor (\frac{n}{dl|A|^{1/d}})^{1/(d-1)} \rfloor$. Notice that under the assumption of Theorem 10.2, (1) stills hold with the new definition of a. We again have two cases:

(I) $\sum_{i=1}^d r_i = 0 (\text{mod } n)$. By the definition of the a_i's, it follows that $\sum_{i=1}^d r_i b_i = 0(\text{mod } n)$ and d should be at least 3. By the definition of the b_i's, it follows immediately that

$$\max_{1 \le i \le d} |r_i| \ge \min(\min_{1 \le j \le d} \frac{b_j}{\sum_{i=1}^{j-1} b_i}, \frac{n}{\sum_{j=1}^d b_j}) \ge \frac{1}{2} a^{1/(d-1)} \ge 2l|A|^{1/d}, \tag{59}$$

where the last inequality is from (1).

(II) $\sum_{i=1}^d r_i \ne 0(\text{mod } n)$. In this case, we have

$$\sum_{j=1}^d r_j a + \sum_{j=1}^d r_j b_j = pn$$

for some integer p. If $p = 0$, then

$$\max_{1 \le i \le d} |r_i| \ge \frac{a}{\sum_{i=1}^d b_i} \ge \frac{1}{2} a^{1/(d-1)} \ge 2l|A|^{1/d}. \tag{60}$$

If $p \ne 0$, then

$$\max_{1 \le i \le d} |r_i| \ge \frac{n}{da + \sum_{j=1}^d b_j} \ge \frac{n}{(d+1)a} \ge \frac{1}{2} a^{1/(d-1)} \ge 2l|A|^{1/d}. \tag{61}$$

Without any further explanation, we now state the analogues of Theorems 5.1, 7.1 and 8.13.

Theorem 10.3. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let $A_1, \dots, A_l$ be sets of residue classes modulo n of size $|A|$ where l and $|A|$ satisfy $l^{d+1}|A| \ge Cn$. Then $A_1 + \cdots + A_l$ either contains all residue classes modulo n or contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer $1 \le d' \le d$.*

Theorem 10.4. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let n be a prime, let l be a positive integer, and let A be a set of residues modulo n such that $l^{d+1}|A| \ge Cn$. Then lA either contains all residue classes modulo n or contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer $1 \le d' \le d$.*

Theorem 10.5. *For any fixed positive integer d there are positive constants C and c depending on d such that the following holds. Let n be a prime, let l be a positive integer, and let $A_1, \dots, A_l$ be sets of residues modulo n such that $|A_1| = \cdots = |A_l| = |A|$ and $l^{d+1}|A| \ge Cn$. Then $A_1 \overset{*}{+} \cdots \overset{*}{+} A_l$ either contains all residue* *classes modulo n or it contains a proper GAP of rank d' and volume at least $cl^{d'}|A|$, for some integer $1 \leq d' \leq d$.*

10.6. **An application.** A set A of residues modulo n is called *zero-sum-free* if none of the subset of A adds up to zero modulo n. Zero-sum-free sets are objects of considerable interest in additive number theory (see Section C of [17] and the references therein). Here we address the following basic question:

How many zero-sum-free sets are there?

We denote by S_A the collection of partial sums of A, so A is zero-sum-free if and only if $0 \notin A$. Szemerédi [26] and Olson [22], answering a question of Erdös, proved that a zero-sum-free set has at most $2n^{1/2}$ elements. This implies that the number of zero-sum-free sets is at most

$$\sum_{i=1}^{\lfloor 2n^{1/2} \rfloor} \binom{n}{i} = 2^{\Omega(n^{1/2}\log_2 n)}.$$

It is not hard to give a lower bound of $2^{\Omega(\sqrt{n})}$; notice that every subset of the interval $[\lfloor\sqrt{2n}-1\rfloor]$ is zero-sum-free, since

$$1 + 2 + \cdots + [\lfloor\sqrt{2n}-1\rfloor] < n.$$

The number of subsets of the above interval is clearly $2^{\Omega(\sqrt{n})}$.

In an earlier paper [27], we succeeded in establishing a sharp bound, using a weaker version of Theorem 10.5. (To be more precise, what we actually used was a weaker version of the finite field analogue of Theorem 3.8.)

Theorem 10.7. *Let n be a prime. The number of zero-sum-free sets* (mod n) *is*

$$2^{(\sqrt{\frac{1}{3}}\pi\log_2 e + o(1))\sqrt{n}}.$$

This surprising estimate might deserve an explanation. To reveals its origin, let us give a short proof for the lower bound. We call a set A of positive integers *n-small* if the sum of the elements in A is less than n. It is trivial that an n-small set is zero-sum-free. On the other hand, the number of n-small sets is $2^{(\sqrt{\frac{1}{3}}\pi\log_2 e + o(1))\sqrt{n}}$ due to the following lemma, which is a well-known result in the theory of partitions (see, for instance, Theorem 6.7 in [1]).

Lemma 10.8. *The number of representations of n as a sum of different positive integers is $2^{(\sqrt{\frac{1}{3}}\pi\log_2 e + o(1))\sqrt{n}}$. Consequently, the number of n-small sets is*

$$2^{(\sqrt{\frac{1}{3}}\pi\log_2 e + o(1))\sqrt{n}}.$$

The hard part of Theorem 10.7 is the upper bound. Using our results on long arithmetic progressions (modulo n), we managed to show that if A is zero-sum-free and has relatively many elements (the number of sets with at most $n^{1/2}/\log_2^2 n$ elements is $2^{o(n^{1/2})}$ so we can ignore these sets), then A is close to be n-small (for the exact statement, see [27]). The general idea is as follows. Let A' be a relatively small subset of A; our results show that $S_{A'}$ contains a rather long arithmetic progression. We next make many translations of this arithmetic progression by adding to it elements from $A\backslash A'$. If all these translations avoid 0, then we have a good chance of deducing a structural property of A, and it turned out that typically A should look like an n-small set. A similar argument can be applied to determine the number of x-sum-free sets, for any non-zero residue class x. Trying not to spoil the fun, we do not state the theorem here (it can be found in [27]), but let us mention that the bound for non-zero x is different from the bound in Theorem 10.7. Guessing this bound is a good puzzle that the reader who bears with us until the very end might enjoy.

References

[1] G. Andrews, The theory of partitions. Reprint of the 1976 original. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998. MR1634067 (99c:11126)

[2] Y. Bilu, Structure of sets with small sumset. Structure theory of set addition, *Astérisque*, No. 258 (1999), xi, 77–108. MR1701189 (2000h:11109)

[3] J. Bourgain, On arithmetic progressions in sums of sets of integers. A tribute to Paul Erdös, pp. 105–109, Cambridge Univ. Press, Cambridge, 1990. MR1117007 (92e:11011)

[4] J. W. S. Cassels, On the representation of integers as the sums of distinct summands taken from a fixed set, *Acta Sci. Math. Szeged* 21 (1960), 111–124. MR0130236 (24:A103)

[5] M-C. Chang, A polynomial bound in Freiman's theorem, *Duke Math. J.* 113 (2002), 399–419. MR1909605 (2003d:11151)

[6] Y. G. Chen, On subset sums of a fixed set, *Acta Arith.* 106 (3) (2003), 207–211. MR1957105 (2003j:11019)

[7] D. da Silva and Y. O. Hamidoune, Cyclic Spaces for Grassmann Derivatives and Additive Theory, *Bull. London Math. Soc.* 26 (1994), 140–146. MR1272299 (95i:11007)

[8] P. Erdös, On the representation of large integers as sums of distinct summands taken from a fixed set, *Acta. Arith.* 7 (1962), 345-354. MR0144846 (26:2387)

[9] P. Erdös and R. Graham, Old and new problems and results in combinatorial number theory. Monographies de L'Enseignement Mathématique 28. Université de Genève, L'Enseignement Mathématique, Geneva, 1980. MR0592420 (82j:10001)

[10] G. Freiman, H. Halberstam, and I. Ruzsa, Integer sum sets containing long arithmetic progressions, *J. London Math. Soc.* (2) 46 (1992), no. 2, 193–201. MR1182477 (93j:11008)

[11] G. Freiman, New analytical results in subset-sum problem. Combinatorics and algorithms (Jerusalem, 1988). *Discrete Math.* 114 (1993), no. 1-3, 205–217. MR1217753 (94b:11013)

[12] G. Freiman, Foundations of a structural theory of set addition. Translated from the Russian. *Translations of Mathematical Monographs,* Vol. 37, American Mathematical Society, Providence, R. I., 1973. vii+108 pp. MR0360496 (50:12944)

[13] G. Freiman, Structure theory of set addition, *Astérisque* 258 (1999), xi, 1–33. MR1701187 (2000j:11147)

[14] J. Folkman, On the representation of integers as sums of distinct terms from a fixed sequence, *Canad. J. Math.* 18 (1966), 643–655. MR0199169 (33:7318)

[15] R. Graham, Complete sequences of polynomial values, *Duke Math. J.* 31 (1964), 275–286. MR0162759 (29:63)

[16] B. Green, Arithmetic progressions in sumsets, *Geom. Funct. Anal.* 12 (2002), no. 3, 584–597. MR1924373 (2003i:11148)

[17] R. Guy, Unsolved problems in Number Theory, Second Edition, Springer-Verlag 1994. MR1299330 (96e:11002)

[18] N. Hegyvári, On the representation of integers as sums of distinct terms from a fixed set, *Acta Arith.* 92 (2000), no. 2, 99–104. MR1750309 (2001c:11014)

[19] V. Lev and P. Smeliansky, On addition of two distinct sets of integers, *Acta Arithmetica* 70 (1) (1995), 85–91. MR1318763 (96f:11035)

[20] V. Lev, Optimal representations by sumsets and subset sums, *Journal of Number Theory* 62 (1) (1997), 127–143. MR1430006 (97k:11015)

[21] T. Łuczak and T. Schoen, On the maximal density of sum-free sets, *Acta Arith.* 95 (2000), no. 3, 225–229. MR1793162 (2001k:11018)

[22] J. Olsen, An addition theorem modulo p, *Journal of Combin. Theory* 5 (1968), 53-58. MR0227130 (37:2715)

[23] C. Pomerance and A. Sárközy, Combinatorial number theory, Handbook of combinatorics, Vol. 1, 2, pp. 967–1018, Elsevier, Amsterdam, 1995. MR1373676 (97e:11032)

[24] I. Ruzsa, Generalized arithmetical progressions and sumsets, *Acta Math. Hungar.* 65 (1994), no. 4, 379–388. MR1281447 (95k:11011)

[25] A. Sárközy, Finite addition theorems I, *J. Number Theory* 32 (1989), 114–130. MR1002119 (91f:11007)

[26] E. Szemerédi, On a conjecture of Erdös and Heilbronn, *Acta Arith.* 17 (1970), 227–229. MR0268159 (42:3058)

[27] E. Szemerédi and V. H. Vu, Long arithmetic progressions in sumsets and the number of x-sum-free sets, *Proceedings of London Mathematics Society* 90 (2005), 273-296.

[28] E. Szemerédi and V. H. Vu, Finite and infinite arithmetic progressions in sumsets, *to appear in Annals of Mathematics.*

[29] R. Vaughan, The Hardy-Littlewood method. Second edition. Cambridge Tracts in Mathematics, 125. Cambridge University Press, Cambridge, 1997. MR1435742 (98a:11133)

[30] V. H. Vu, Olson's theorem for cyclic groups, *submitted.*

[31] V. H. Vu, New results concerning subset sums, *in preparation.*

Department of Computer Science, Rutgers University, New Brunswick, New Jersey 08854

E-mail address: szemered@cs.rutgers.edu

Department of Mathematics, University of California, San Diego, La Jolla, California 92093-0112

E-mail address: vanvu@ucsd.edu

E-mail address: vanvu@math.rutgers.edu