POSITIVITY IN T-EQUIVARIANT K-THEORY OF FLAG VARIETIES ASSOCIATED TO KAC-MOODY GROUPS II

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ABSTRACT. We prove sign-alternation of the structure constants in the basis of the structure sheaves of opposite Schubert varieties in the torus-equivariant Grothendieck group of coherent sheaves on the flag varieties G/P associated to an arbitrary symmetrizable Kac-Moody group G, where P is any parabolic subgroup. This generalizes the work of Anderson-Griffeth-Miller from the finite case to the general Kac-Moody case, and affirmatively answers a conjecture of Lam-Schilling-Shimozono regarding the signs of the structure constants in the case of the affine Grassmannian.

1. INTRODUCTION

Let G be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots and $G^{\min} \subset G$ the minimal Kac-Moody group as in [Ku2, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus, and W the Weyl group [Ku2, Chapter 6]. Let $\bar{X} = G/B$ be the 'thick' flag variety (introduced by Kashiwara) which contains the standard flag variety $X = G^{\min}/B$. Let T be the adjoint torus, i.e., $T := H/Z(G^{\min})$, where $Z(G^{\min})$ denotes the center of $G^{\min}$, and let $R(T)$ denote the representation ring of T. For any $w \in W$, we have the Schubert cell $C_w := BwB/B \subset X$, the Schubert variety $X_w := \overline{C_w} \subset X$, the opposite Schubert cell $C^w := B^-wB/B \subset \bar{X}$, and the opposite Schubert variety $X^w := \overline{C^w} \subset \bar{X}$. When G is a (finite dimensional) semisimple group, it is referred to as the *finite case*.

Let $K_T^0(\bar{X})$ denote the Grothendieck group of T-equivariant coherent sheaves on $\bar{X}$. Then, $\{[\mathcal{O}_{X^w}]\}_{w \in W}$ forms an $R(T)$-'basis' of $K_T^0(\bar{X})$ (where infinite sums are allowed), i.e., $K_T^0(\bar{X}) = \prod_{w \in W} R(T)[\mathcal{O}_{X^w}]$. We express the product in $K_T^0(\bar{X})$ by:

$$[\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}] = \sum_{w \in W} d_{u,v}^w [\mathcal{O}_{X^w}], \text{ for unique } d_{u,v}^w \in R(T).$$

The following result is our main theorem (cf. Theorem 5.3). This was conjectured first by Griffeth-Ram [GR] in the finite case (2004), proven in the finite case by Anderson-Griffeth-Miller [AGM] (2011), and then conjectured in the general Kac-Moody case by Kumar [Ku1] (2012).

Theorem 1.1. *For any* $u, v, w \in W$,

$$(-1)^{\ell(w)+\ell(u)+\ell(v)} d_{u,v}^w \in \mathbb{Z}_{\geq 0}[(e^{-\alpha_1} - 1), \ldots, (e^{-\alpha_r} - 1)],$$

where $\{\alpha_1 \ldots, \alpha_r\}$ *are the simple roots.*

Let P be any standard parabolic subgroup of G of finite type (cf. [Ku2, Definition 6.1.18]). (Recall that a parabolic subgroup is said to be of *finite type* if its Levi subgroup is finite dimensional.) We may express, in $K_T^0(G/P)$,

$$[\mathcal{O}_{X_P^u}] \cdot [\mathcal{O}_{X_P^v}] = \sum_{w \in W^P} d_{u,v}^w(P)[\mathcal{O}_{X_P^w}], \text{ for unique } d_{u,v}^w(P) \in R(T),$$

where W^P is the set of <u>minimal length</u> representatives of W/W_P, W_P is the Weyl group of P, and $X_P^w := \overline{B^- wP/P} \subset G/P$ is the opposite Schubert variety.

Let $\pi : G/B \to G/P$ be the standard (T-equivariant) projection. Then, π is a locally trivial fibration (with fiber the smooth projective variety P/B) and hence flat (cf. [Ku2, Chapter 7]). Thus, we have

$$\pi^*[\mathcal{O}_{X_P^w}] = [\mathcal{O}_{\pi^{-1}(X_P^w)}] = [\mathcal{O}_{X^w}].$$

Since $\pi^* : K_T^0(G/P) \to K_T^0(G/B)$ is a ring homomorphism, we have $d_{u,v}^w = d_{u,v}^w(P)$ for any $u, v, w \in W^P$ and hence Theorem 1.1 immediately generalizes from the case of G/B to the case of G/P, and we obtain:

Theorem 1.2. *For any standard parabolic subgroup P of G of finite type, and any $u, v, w \in W^P$,*

$$(-1)^{\ell(w)+\ell(u)+\ell(v)} d_{u,v}^w(P) \in \mathbb{Z}_{\geq 0}[(e^{-\alpha_1} - 1), \ldots, (e^{-\alpha_r} - 1)].$$

Theorems 1.1 and 1.2 also apply to ordinary (non-equivariant) K-theory. Let $K^0(\bar{X})$ denote the Grothendieck group of coherent sheaves on $\bar{X}$. Then, we have $K^0(\bar{X}) = \prod_{w \in W} \mathbb{Z}[\mathcal{O}_{X^w}]$. Further, the map

$$\mathbb{Z} \otimes_{R(T)} K_T^0(\bar{X}) \to K^0(\bar{X}), 1 \otimes [\mathcal{O}_{X^w}] \mapsto [\mathcal{O}_{X^w}],$$

is an isomorphism, where we view $\mathbb{Z}$ as an $R(T)$-module via evaluation at 1. Similar results apply to G/P.

Write, in $K^0(G/P)$, for $u, v \in W^P$,

$$[\mathcal{O}_{X_P^u}] \cdot [\mathcal{O}_{X_P^v}] = \sum_{w \in W^P} a_{u,v}^w(P)[\mathcal{O}_{X_P^w}], \text{ for unique } a_{u,v}^w(P) \in \mathbb{Z}.$$

Then, by the above, along with Theorem 1.2 we have:

Theorem 1.3. *For any standard parabolic subgroup P of G of finite type, and any $u, v, w \in W^P$,*

$$(-1)^{\ell(w)+\ell(u)+\ell(v)} a_{u,v}^w(P) \in \mathbb{Z}_{\geq 0}.$$

The following conjecture of Lam-Schilling-Shimozono [LSS, Conjectures 7.20 (ii) and 7.21 (iii)] is a special case of Theorem 1.3:

Let $G = \widehat{SL_N}$ be the affine Kac-Moody group associated to SL_N, and let P be its standard maximal parabolic subgroup. Let $\bar{\mathcal{X}} = G/P$ be the corresponding infinite Grassmannian. Then, $K^0(\bar{\mathcal{X}})$ has the structure sheaf 'basis' $\{[\mathcal{O}_{X^u}]\}_{u \in W^P}$ over $\mathbb{Z}$, where W is the (affine) Weyl group of G and W^P is the set of minimal coset representatives in W/W_o (W_o being the finite Weyl group of SL_N). Write, for any $u, v \in W^P$,

$$[\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}] = \sum_{w \in W^P} b_{u,v}^w[\mathcal{O}_{X^w}], \text{ for unique integers } b_{u,v}^w.$$

Then, the following is a special case of Theorem 1.3:

Corollary 1.4 (Conjectured by Lam-Schilling-Shimozono)**.**

$$(-1)^{\ell(u)+\ell(v)+\ell(w)} b_{u,v}^w \in \mathbb{Z}_{\geq 0}.$$

The proof of Theorem 1.1 follows closely the work of Anderson-Griffeth-Miller [AGM] and Kumar [Ku1]; *though several technical details had to be carefully addressed, e.g., Proposition 7.7 proving the rational singularities of $\mathcal{Z}$.* Letting $K_0^T(X)$ denote the Grothendieck group of finitely supported T-equivariant coherent sheaves on X, there is a natural pairing

$$\langle \ , \ \rangle : K_T^0(\bar{X}) \otimes K_0^T(X) \to R(T),$$

coming from the T-equivariant Euler-Poincaré characteristic (cf. (1)). Under this pairing, the bases $\{[\mathcal{O}_{X^u}]\}$ of $K_T^0(\bar{X})$ and $\{[\xi_w]\}$ of $K_0^T(X)$ are dual (cf. Proposition 3.10), where $\xi_w := \mathcal{O}_{X_w}(-\partial X_w)$ and $\partial X_w := X_w \backslash C_w$. (By Lemma 3.5, ξ_w is indeed a basis of $K_0^T(X)$.)

Then, realizing the product structure constants in $K_T^0(\bar{X})$ as the coproduct structure constants of the dual basis in $K_0^T(X)$ (cf. Lemma 4.2) allows the use of the above pairing and duality to express the structure constants in terms of certain cohomology groups. Following [AGM], we introduce the 'mixing space' $X_\mathbb{P}$, which is a bundle over a product of projective spaces $\mathbb{P}$ with fiber X. This allows for the reduction from T-equivariant K-theory to non-equivariant K-theory. Then, we introduce the 'mixing group' Γ (cf. Definition 4.11) whose action is sufficient to ensure a certain transverality needed to prove our main result.

Theorem 5.1 is our main technical result. Part a) allows the structure constants to be realized as the Euler characteristic of a certain sheaf, while part b) shows that this sheaf has cohomology which is zero in all but a single term of known degree, immediately resulting in a determination of the sign of the Euler characteristic and hence the structure constants.

Part a) of Theorem 5.1 is proved in Section 6. The proof relies on some local $\mathscr{T}or$ vanishing results which were proven in [Ku1], as well as a reduction to finite dimensional schemes, where a transversality result due to Sierra is crucially used (cf. Theorem 6.1).

The proof of the more difficult part b) of Theorem 5.1 is the content of Sections 7 through 9. In Section 7 we introduce the crucial scheme $\mathcal{Z}$ which comes with a projection to the mixing group Γ. We also introduce a desingularization $\widetilde{\mathcal{Z}}$ of $\mathcal{Z}$ and prove that $\mathcal{Z}$ is irreducible, normal and has rational singularities (cf. Propositions 7.5 and 7.7). We further introduce the divisor $\partial \mathcal{Z} \subset \mathcal{Z}$ which is shown to be Cohen-Macaulay (cf. Proposition 7.8). It is on the fibers of the projection $\pi : \mathcal{Z} \to \Gamma$ that the computation of the cohomology of the previously mentioned sheaf occurs.

In Section 8 the rational singularities of $\mathcal{Z}$ are used to apply a relative version of the Kawamata-Viehweg vanishing theorem (cf. Theorem 8.1) to show that $R^i \pi_* \omega_{\mathcal{Z}}(\partial \mathcal{Z}) = 0$ for all $i > 0$ (cf. Corollary 8.8). Finally, in Section 9, this vanishing of the higher direct images along with the semicontinuity theorem is used to prove vanishing of the relevant cohomology along the fibers of $\pi : \mathcal{Z} \to \Gamma$ (cf. Theorem 9.2) and thus conclude the proof of part b) of Theorem 5.1.

2. Notation

We work over the field $\mathbb{C}$ of complex numbers. By a variety we mean an algebraic variety over $\mathbb{C}$ which is reduced, but not necessarily irreducible.

Let G be any symmetrizable Kac-Moody group over $\mathbb{C}$ completed along the negative roots (as opposed to completed along the positive roots as in [Ku2, Chapter 6]) and $G^{\min} \subset G$ the minimal Kac-Moody group as in [Ku2, §7.4]. Let B be the standard (positive) Borel subgroup, B^- the standard negative Borel subgroup, $H = B \cap B^-$ the standard maximal torus, and W the Weyl group [Ku2, Chapter 6]. Let

$$\bar{X} = G/B$$

be the 'thick' flag variety which contains the standard flag variety $X = G^{\min}/B$.

If G is not of finite type, $\bar{X}$ is an infinite dimensional non-quasi-compact scheme [K, §4] and X is an ind-projective variety [Ku2, §7.1]. The group $G^{\min}$ and, in particular, the maximal torus H acts on $\bar{X}$ and X.

Let T be the adjoint torus, i.e., $T := H/Z(G^{\min})$, where $Z(G^{\min})$ denotes the center of $G^{\min}$. (Recall that by [Ku2, Lemma 6.2.9(c)], $Z(G^{\min}) = \{h \in H : e^{\alpha_i}(h) = 1$ for all the simple roots $\alpha_i\}$.) Then, the action of H on $\bar{X}$ (and on X) descends to an action of T.

For any $w \in W$, we have the Schubert cell

$$C_w := BwB/B \subset X,$$

the Schubert variety

$$X_w := \overline{C_w} = \bigsqcup_{w' \leq w} C_{w'} \subset X,$$

the opposite Schubert cell

$$C^w := B^- wB/B \subset \bar{X},$$

and the opposite Schubert variety

$$X^w := \overline{C^w} = \bigsqcup_{w' \geq w} C^{w'} \subset \bar{X},$$

all endowed with the reduced subscheme structures. Then, X_w is a (finite dimensional) irreducible projective subvariety of X and X^w is a finite codimensional irreducible subscheme of $\bar{X}$ [Ku2, §7.1] and [K, §4]. We denote by Z_w the Bott-Samelson-Demazure-Hansen (BSDH) variety as in [Ku2, §7.1.3], which is a B-equivariant desingularization of X_w [Ku2, Proposition 7.1.15]. Further, X_w is normal, Cohen-Macaulay (CM for short) and has rational singularities [Ku2, Theorem 8.2.2].

We also define the boundary of the Schubert variety by

$$\partial X_w := X_w \setminus C_w$$

with the reduced subscheme structure. Then, ∂X_w is pure of codimension 1 in X_w and is CM (See the proof of Proposition 7.8).

For any $u \leq w$, we have the Richardson variety

$$X_w^u := X^u \cap X_w \subset X$$

endowed with the reduced subvariety structure. By [Ku1, Proposition 6.6], X_w^u is irreducible, normal and CM. We denote by Z_w^u the T-equivariant desingularization of X_w^u as in [Ku1, Theorem 6.8]. By [KuS, Theorem 3.1], X_w^u has rational singularities (in fact it has Kawamata log terminal singularities).

We denote the representation ring of T by $R(T)$. Let $\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ be the set of simple roots, $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \mathfrak{h}$ the set of simple coroots, and $\{s_1, \ldots, s_r\}$ the corresponding set of simple reflections, where $\mathfrak{h} := \mathrm{Lie}(H)$. Let $\rho \in \mathfrak{h}^*$ be any integral weight satisfying

$$\rho(\alpha_i^\vee) = 1, \text{ for all } 1 \le i \le r.$$

When G is a finite dimensional semisimple group, ρ is unique, but for a general Kac-Moody group G, it may not be unique.

For any integral weight λ let $\mathbb{C}_\lambda$ denote the one-dimensional representation of H on $\mathbb{C}$ given by $h \cdot v = \lambda(h)v$ for $h \in H, v \in \mathbb{C}$. By extending this action to B we may define, for any integral weight λ, the G-equivariant line bundle $\mathcal{L}(\lambda)$ on $\bar{X}$ by

$$\mathcal{L}(\lambda) := G \times^B \mathbb{C}_{-\lambda},$$

where, for any representation V of B, $G \times^B V := (G \times V)/B$ and B acts on $G \times V$ via $(g, v) \cdot b = (gb, b^{-1}v)$ for $g \in G, v \in V, b \in B$. Then, $G \times^B V$ is the total space of a G-equivariant vector bundle over X, with projection given by $(g, v)B \mapsto gB$. We also define the bundle

$$e^\lambda := \bar{X} \times \mathbb{C}_\lambda,$$

which while trivial when viewed as a non-equivariant line bundle, is equivariantly non-trivial with the diagonal action of H.

3. The Grothendieck group

For a quasi-compact scheme Y, an $\mathcal{O}_Y$-module is called *coherent* if it is finitely presented as an $\mathcal{O}_Y$-module and any $\mathcal{O}_Y$-submodule of finite type admits a finite presentation.

A subset $S \subset W$ is called an *ideal* if for all $x \in S$ and $y \le x$ we have $y \in S$. We say that an $\mathcal{O}_{\bar{X}}$-module $\mathcal{S}$ is *coherent* if $\mathcal{S}|_{V^S}$ is a coherent $\mathcal{O}_{V^S}$-module for every finite ideal $S \subset W$, where V^S is the quasi-compact open subset of $\bar{X}$ defined by

$$V^S := \bigcup_{w \in S} wU^- B/B, \text{ where } U^- := [B^-, B^-].$$

Let $K_T^0(\bar{X})$ denote the Grothendieck group of T-equivariant coherent $\mathcal{O}_{\bar{X}}$-modules. Since the coherence condition on $\mathcal{S}$ is imposed only for $\mathcal{S}|_{V^S}$ for finite ideals $S \subset W$, $K_T^0(\bar{X})$ can be thought of as the inverse limit of $K_T^0(V^S)$ as S varies over all finite ideals of W (cf. [KS, §2]).

We define

$$K_0^T(X) := \mathrm{Limit}_{n \to \infty} K_0^T(X_n),$$

where $\{X_n\}_{n \ge 1}$ is the filtration of X giving the ind-projective variety structure and $K_0^T(X_n)$ is the Grothendieck group of T-equivariant coherent sheaves on X_n.

For any $u \in W$, $\mathcal{O}_{X^u}$ is a coherent $\mathcal{O}_{\bar{X}}$-module by [KS, §2]. From [KS, comment after Remark 2.4] we have:

Lemma 3.1. $\{[\mathcal{O}_{X^u}]\}$ *forms a basis of $K_T^0(\bar{X})$ as an $R(T)$-module (where we allow arbitrary infinite sums).*

Lemma 3.2. $\{[\mathcal{O}_{X_w}]\}$ *forms a basis of $K_0^T(X)$ as an $R(T)$-module.*

Proof. This follows from [CG, §5.2.14 and Theorem 5.4.17]. $\square$

The following lemma is due to Kashiwara-Shimozono [KS, Lemma 8.1].

Lemma 3.3. *Any T-equivariant coherent sheaf $\mathcal{S}$ on V^u admits a free resolution in $\mathrm{Coh}_T(\mathcal{O}_{V^u})$:*

$$0 \to S_n \otimes \mathcal{O}_{V^u} \to \cdots \to S_1 \otimes \mathcal{O}_{V^u} \to S_0 \otimes \mathcal{O}_{V^u} \to \mathcal{S} \to 0,$$

where S_k are finite dimensional T-modules, $V^u := uU^- B/B \subset \bar{X}$, and $\mathrm{Coh}_T(\mathcal{O}_{V^u})$ denotes the abelian category of T-equivariant coherent $\mathcal{O}_{V^u}$-modules.

We define a pairing

(1) $\langle \ , \ \rangle : K_T^0(\bar{X}) \otimes K_0^T(X) \to R(T),$

$$\langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi_T(X_n, \mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{S}, \mathcal{F})),$$

where $\mathcal{S}$ is a T-equivariant coherent sheaf on $\bar{X}$ and $\mathcal{F}$ is a T-equivariant coherent sheaf on X supported on X_n for some n, where χ_T represents the T-equivariant Euler-Poincaré characteristic. By [Ku1, Lemma 3.5] this is well defined.

Definition 3.4. We define ξ_w to be the ideal sheaf of ∂X_w in X_w, where ∂X_w is given the reduced subscheme structure:

$$\xi_w := \mathcal{O}_{X_w}(-\partial X_w).$$

Lemma 3.5. *$\{[\xi_w]\}$ forms a basis of $K_0^T(X)$ as an $R(T)$-module.*

Proof. This follows since $[\xi_w] = [\mathcal{O}_{X_w}] + \sum_{w' < w} r_{w'}[\mathcal{O}_{X_{w'}}]$, for some $r_{w'} \in R(T)$ and the fact that $[\mathcal{O}_{X_w}]$ is a basis of $K_0^T(X)$ (cf. Lemma 3.2). $\square$

Proposition 3.6. *$\omega_{X_w} = e^{-\rho}\mathscr{L}(-\rho)\xi_w$, where ω_{X_w} is the dualizing sheaf of X_w.*

Proof. This follows from [GK, Proposition 2.2] since the same proof works for general Kac-Moody groups. $\square$

From [Ku1, Lemma 5.5] we have the following result:

Lemma 3.7. *For any $u, w \in W$, we have*

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{X_w}) = 0, \ \forall i > 0.$$

We now prove:

Lemma 3.8. *For any $u \in W$ and any finite union of Schubert varieties $Y = \bigcup_{i=1}^{\ell} X_{w_i}$ we have*

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_Y) = 0, \ \forall i > 0.$$

In particular, for any $u, w \in W$ we have

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{\partial X_w}) = 0, \ \forall i > 0.$$

Proof. We proceed by double induction on the dimension of Y (i.e., the largest dimension of the irreducible components of Y) and the number of irreducible components of Y. If $\dim Y = 0$, then $Y = X_e$, and so the result follows from Lemma 3.7. Now, suppose that $\dim Y = d$ and Y has k irreducible components. If $k = 1$, then the result follows from Lemma 3.7, so we may assume that $k \geq 2$. Let $Y_1 = X_w$ be an irreducible component of Y and let Y_2 be the union of all the other irreducible components.

By [KuS, Proposition 5.3 and its proof] X is Frobenius split compatibly splitting its Schubert varieties X_u and also Richardson varieties X_u^v.

($*$)   Thus, any finite intersection

$$X_{u_1} \cap \ldots \cap X_{u_n} \cap X^{v_1} \cap \ldots \cap X^{v_m} \text{ is reduced for any } n \geq 1.$$

(In the proof here we have only used $X_{u_1} \cap \ldots \cap X_{u_n}$ to be reduced, but the more general assertion here will be used later in the paper.)

The short exact sequence of sheaves

$$0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2} \to 0$$

yields the long exact sequence

$$\ldots \to \mathscr{T}or_{i+1}^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{Y_1 \cap Y_2}) \to \mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_Y)$$
$$\to \mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2}) \to \ldots.$$

Now, since Y_2 has less than k irreducible components, induction on the number of irreducible components gives

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2}) = 0, \ \forall i > 0.$$

Since $Y_1 \cap Y_2$ is reduced and of dimension less than d, induction on the dimension gives

$$\mathscr{T}or_{i+1}^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \mathcal{O}_{Y_1 \cap Y_2}) = 0, \ \forall i > 0.$$

Together, these imply the lemma. $\square$

Lemma 3.9. *For any $u, w \in W$, we have*

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \xi_w) = 0, \ \forall i > 0.$$

Proof. Applying Lemmas 3.7 and 3.8, the desired result follows from the long exact sequence for $\mathscr{T}or$. $\square$

Proposition 3.10. *For any $u, w \in W$ we have*

$$\langle [\mathcal{O}_{X^u}], [\xi_w] \rangle = \delta_{u,w}.$$

Proof. By definition

$$\langle [\mathcal{O}_{X^u}], [\xi_w] \rangle = \sum_i (-1)^i \chi_T(X_n, \mathscr{T}or_i^{\mathcal{O}_{\bar{X}}}(\mathcal{O}_{X^u}, \xi_w)),$$

where n is taken such that $n \geq \ell(w)$. Thus, by Lemma 3.9, we have

$$\langle [\mathcal{O}_{X^u}], [\xi_w] \rangle = \chi_T(X_n, \mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \xi_w).$$

By Lemma 3.8 we have the sheaf exact sequence

$$0 \to \mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \xi_w \to \mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} \to \mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\partial X_w} \to 0.$$

Observe that by ($*$)

$$\mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{X_w} = \mathcal{O}_{X^u \cap X_w},$$

and similarly for $\mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \mathcal{O}_{\partial X_w}$. Thus,

$$\chi_T(X_n, \mathcal{O}_{X^u} \otimes_{\mathcal{O}_{\bar{X}}} \xi_w) = \chi_T(X_n, \mathcal{O}_{X^u \cap X_w}) - \chi_T(X_n, \mathcal{O}_{X^u \cap \partial X_w}).$$

When non-empty, $X^u \cap X_w$ is irreducible by [Ku1, Proposition 6.6], and thus $X^u \cap \partial X_w = \bigcup_{u \le w' < w} X^u \cap X_{w'}$ is a connected projective variety when non-empty, since $u \in X^u \cap \bar{X}_{w'}$ for all $u \le w' < w$.

If $u \not\le w$ we have $X^u \cap X_w = \emptyset$, so assume $u \le w$. In this case $X^u \cap X_w \neq \emptyset$. Now, if $u = w$, then $X^u \cap \partial X_w = \emptyset$, while if $u < w$, then $X^u \cap \partial X_w \neq \emptyset$. By [KuS, Corollary 3.2],

$$H^i(X_n, \mathcal{O}_{X^u \cap X_w}) = 0, \ \forall i > 0.$$

Using an inductive argument similar to Lemma 3.8 we obtain

$$H^i(X_n, \mathcal{O}_{X^u \cap Y}) = 0, \ \forall i > 0,$$

where Y is any finite union of Schubert varieties. Taking $Y = X^u \cap \partial X_w$ and combining the above implies the proposition (here we use that, when non-empty, $X^u \cap Y$ is connected). $\qquad\square$

4. The mixing space and mixing group

In this section we realize the product structure constants of $\{[\mathcal{O}_{X^u}]\}$ in $K_T^0(\bar{X})$ as the coproduct structure constants of the dual basis $\{[\xi_u]\}$ in $K_0^T(X)$ (Lemma 4.2). We then introduce the mixing space $X_{\mathbb{P}}$, which is a bundle over a product of projective spaces $\mathbb{P}$, with fiber X. This allows for the reduction from T-equivariant K-theory to non-equivariant K-theory. Using the pairing and duality introduced in the previous section, we realize the structure constants in terms of certain cohomology groups (cf. Lemma 4.10). Finally, we introduce the mixing group Γ whose action is sufficient to allow for a transversality result necessary to prove part a) of our main technical result (Theorem 5.1).

Definition 4.1 (Structure constants $d_{u,v}^w$)**.** By Lemma 3.1, in $K_T^0(\bar{X})$ we have:

$$(2) \qquad [\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}] = \sum_{w \in W} d_{u,v}^w [\mathcal{O}_{X^w}], \ \text{for unique } d_{u,v}^w \in R(T),$$

where infinitely many of $d_{u,v}^w$ may be non-zero.

Lemma 4.2. *Write, in $K_0^T(X \times X)$ under the diagonal action of T on $X \times X$,*

$$(3) \qquad \Delta_*[\xi_w] = \sum_{u,v} e_{u,v}^w [\xi_u \boxtimes \xi_v], \ \text{for } e_{u,v}^w \in R(T),$$

where $\Delta : X \to X \times X$ is the diagonal map. Then, $e_{u,v}^w = d_{u,v}^w$.

Proof. Let $\bar{\Delta} : \bar{X} \to \bar{X} \times \bar{X}$ be the diagonal map, and note that $\bar{\Delta}^*[\mathcal{O}_{X^u} \boxtimes \mathcal{O}_{X^v}] = [\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}]$. Hence, we have:

$$d^w_{u,v} = \langle \sum_{w' \in W} d^{w'}_{u,v} [\mathcal{O}_{X^{w'}}], [\xi_w] \rangle, \text{ by Proposition 3.10}$$

$$= \langle [\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}], [\xi_w] \rangle, \text{ by (2)}$$

$$= \langle \bar{\Delta}^*[\mathcal{O}_{X^u} \boxtimes \mathcal{O}_{X^v}], [\xi_w] \rangle$$

$$= \langle [\mathcal{O}_{X^u} \boxtimes \mathcal{O}_{X^v}], \Delta_*[\xi_w] \rangle$$

$$= \langle [\mathcal{O}_{X^u} \boxtimes \mathcal{O}_{X^v}], \sum_{u',v'} e^w_{u',v'} [\xi_{u'} \boxtimes \xi_{v'}] \rangle, \text{ by (3)}$$

$$= e^w_{u,v}, \text{ by Proposition 3.10.}$$

$\square$

We now prepare to define the mixing space. Let $\mathbb{P} := (\mathbb{P}^N)^r$ where $r = \dim T$ and N is some large fixed integer. Let $[N] = \{0, 1, \ldots, N\}$ and let $j = (j_1, j_2, \ldots, j_r) \in [N]^r$. We define

$$\mathbb{P}^j := \mathbb{P}^{N-j_1} \times \cdots \times \mathbb{P}^{N-j_r}$$

and similarly define

$$\mathbb{P}_j := \mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_r}.$$

We also define the boundary of $\mathbb{P}_j$ by

$$\partial \mathbb{P}_j := \left(\mathbb{P}^{j_1-1} \times \mathbb{P}^{j_2} \times \cdots \times \mathbb{P}^{j_r} \right) \cup \cdots \cup \left(\mathbb{P}^{j_1} \times \cdots \times \mathbb{P}^{j_{r-1}} \times \mathbb{P}^{j_r-1} \right),$$

where we interpret $\mathbb{P}^{-1} := \emptyset$ to be the empty set. *Throughout this paper we will identify $T \simeq (\mathbb{C}^*)^r$ via $t \mapsto (e^{-\alpha_1}(t), \ldots, e^{-\alpha_r}(t))$.*

Definition 4.3 (Mixing space $X_{\mathbb{P}}$). Let $E(T)_{\mathbb{P}} := (\mathbb{C}^{N+1} \setminus \{0\})^r$ be the total space of the standard principal T-bundle $E(T)_{\mathbb{P}} \to \mathbb{P}$. We can view $E(T)_{\mathbb{P}} \to \mathbb{P}$ as a finite dimensional approximation of the classifying bundle for T. Define $X_{\mathbb{P}} := E(T)_{\mathbb{P}} \times^T X$ and let $p : X_{\mathbb{P}} \to \mathbb{P}$ be the Zariski-locally trivial fibration with fiber $X = G^{\min}/B$.

For any T-scheme V we define $V_{\mathbb{P}} := E(T)_{\mathbb{P}} \times^T V$.

For the rest of this paper we use the notation $Y := X \times X$ and $\bar{Y} := \bar{X} \times \bar{X}$. Further, we let $Y_{\mathbb{P}} := E(T)_{\mathbb{P}} \times^T Y \simeq X_{\mathbb{P}} \times_{\mathbb{P}} X_{\mathbb{P}}$, where T acts diagonally on Y.

Note that for any $u, v \in W$, $(X_u)_{\mathbb{P}}$ and $(X_u \times X_v)_{\mathbb{P}}$ are CM, as they are fiber bundles over $\mathbb{P}$, and hence locally a product of CM schemes. Thus, they have dualizing sheaves.

Definition 4.4. We define the sheaf $(\xi_u)_{\mathbb{P}}$ on $X_{\mathbb{P}}$ by

$$(\xi_u)_{\mathbb{P}} := (e^\rho \mathscr{L}(\rho))_{\mathbb{P}} \otimes \omega_{(X_u)_{\mathbb{P}}},$$

where $(e^\rho \mathscr{L}(\rho))_{\mathbb{P}}$ is defined by

$$E(T)_{\mathbb{P}} \times^T e^\rho \mathscr{L}(\rho) \to X_{\mathbb{P}}.$$

Proposition 4.5. $K_0(X_{\mathbb{P}}) := \mathrm{Limit}_{n \to \infty} K_0((X_n)_{\mathbb{P}})$ *is a free module over the ring* $K_0(\mathbb{P}) = K^0(\mathbb{P})$ *with basis* $\{[(\xi_w)_{\mathbb{P}}]\}$.

Thus, $K_0(X_{\mathbb{P}})$ *has* $\mathbb{Z}$-*basis*

$$\{p^*([\mathcal{O}_{\mathbb{P}^j}]) \cdot [(\xi_w)_{\mathbb{P}}]\}_{j \in [N]^r, w \in W},$$

where, as above, $p : X_{\mathbb{P}} \to \mathbb{P}$ *is the projection.*

Proof. This follows from [CG, §5.2.14 and Theorem 5.4.17] as well as the fact that the transition matrix between the basis $\{[\mathcal{O}_{X_w}]\}$ and $\{[\xi_w]\}$ is upper triangular with invertible diagonal entries. $\qquad\square$

Definition 4.6. We define the sheaf $\widetilde{\xi_u \boxtimes \xi_v}$ on $Y_{\mathbb{P}}$ by

$$\widetilde{\xi_u \boxtimes \xi_v} := (e^{2\rho}\mathscr{L}(\rho) \boxtimes \mathscr{L}(\rho))_{\mathbb{P}} \otimes \omega_{(X_u \times X_v)_{\mathbb{P}}},$$

where $(e^{2\rho}\mathscr{L}(\rho) \boxtimes \mathscr{L}(\rho))_{\mathbb{P}}$ is defined by

$$E(T)_{\mathbb{P}} \times^T \left(e^{2\rho}\mathscr{L}(\rho) \boxtimes \mathscr{L}(\rho) \right) \to Y_{\mathbb{P}}.$$

The diagonal map $\Delta : X \to Y$ gives rise to the embedding $\tilde{\Delta} : X_{\mathbb{P}} \to Y_{\mathbb{P}}$. Thus, by the previous proposition, we may write

(4)
$$\tilde{\Delta}_*[(\xi_w)_{\mathbb{P}}] = \sum_{u,v \in W, j \in [N]^r} c_{u,v}^w(j)\hat{p}^*[\mathcal{O}_{\mathbb{P}^j}] \cdot [\widetilde{\xi_u \boxtimes \xi_v}] \in K_0(Y_{\mathbb{P}}), \text{ for unique } c_{u,v}^w(j) \in \mathbb{Z},$$

where $\hat{p} : Y_{\mathbb{P}} \to \mathbb{P}$ is the projection.

The following lemma makes precise the reduction from T-equivariant K-theory of $\bar{X}$ to the ordinary K-theory of the mixing space.

Lemma 4.7. *For any* $u, v, w \in W$ *we can choose* N *large enough and express*

$$d_{u,v}^w = \sum_{j \in [N]^r} d_{u,v}^w(j)(e^{-\alpha_1} - 1)^{j_1} \dots (e^{-\alpha_r} - 1)^{j_r},$$

where $d_{u,v}^w(j) \in \mathbb{Z}$.

Then,

$$d_{u,v}^w(j) = (-1)^{|j|}c_{u,v}^w(j), \text{ where } |j| := \sum_{i=1}^{r} j_i.$$

Proof. This follows from Lemma 4.2 and [GK, Lemma 6.2] (see also [AGM, §3]). $\quad\square$

Lemma 4.8. *For any coherent sheaf* $\mathcal{S}$ *on* $\mathbb{P}$ *and any* $u, v \in W$ *we have:*

a) $\mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}}(\bar{p}^*(\mathcal{S}), \widetilde{\xi_u \boxtimes \xi_v}) = 0, \; \forall i > 0,$

b) $\mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}}(\bar{p}^*(\mathcal{S}), \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}) = 0, \; \forall i > 0,$

where $\bar{p} : \bar{Y}_{\mathbb{P}} \to \mathbb{P}$ *is the projection.*

Proof. As the statements are local in $\mathbb{P}$, we may replace $\bar{Y}_{\mathbb{P}}$ by $\mathcal{U} \times \bar{Y}$, for some open set $\mathcal{U} \subset \mathbb{P}$. Then,

$$\bar{p}^*\mathcal{S} \simeq S \boxtimes \mathcal{O}_{\bar{Y}},$$
$$\widetilde{\xi_u \boxtimes \xi_v} \simeq \mathcal{O}_{\mathcal{U}} \boxtimes (\xi_u \boxtimes \xi_v),$$
$$\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \simeq \mathcal{O}_{\mathcal{U}} \boxtimes (\mathcal{O}_{X^u} \boxtimes \mathcal{O}_{X^v}).$$

Applying the above, followed by the Kunneth formula, gives for part a),

$$\mathscr{T}or_i^{\mathcal{O}_{\mathcal{U} \times \bar{Y}}}(\bar{p}^*(\mathcal{S}), \widetilde{\xi_u \boxtimes \xi_v}) = \mathscr{T}or_i^{\mathcal{O}_{\mathcal{U}} \boxtimes \mathcal{O}_{\bar{Y}}}(\mathcal{S} \boxtimes \mathcal{O}_{\bar{Y}}, \mathcal{O}_{\mathcal{U}} \boxtimes (\xi_u \boxtimes \xi_v))$$
$$= \bigoplus_{j+k=i} \mathscr{T}or_j^{\mathcal{O}_{\mathcal{U}}}(\mathcal{S}, \mathcal{O}_{\mathcal{U}}) \otimes \mathscr{T}or_k^{\mathcal{O}_{\bar{Y}}}(\mathcal{O}_{\bar{Y}}, \xi_u \boxtimes \xi_v)$$
$$= 0, \ \text{for } i > 0.$$

A similar computation gives part b).   □

Lemma 4.9. *For any coherent sheaf $\mathcal{S}$ on $\mathbb{P}$ and any $u \in W$ we have:*

$$\mathscr{T}or_i^{\mathcal{O}_{Y_{\mathbb{P}}}}(\hat{p}^*(\mathcal{S}), \tilde{\Delta}_*((\xi_u)_{\mathbb{P}})) = 0, \ \forall i > 0,$$

where, as earlier, $\hat{p} : Y_{\mathbb{P}} \to \mathbb{P}$ is the projection.

Proof. As before, since the statement is local in $\mathbb{P}$, we may replace $Y_{\mathbb{P}}$ by $\mathcal{U} \times Y$, for some open set $\mathcal{U} \subset \mathbb{P}$. Then,

$$\hat{p}^* S \simeq S \boxtimes \mathcal{O}_Y,$$
$$\tilde{\Delta}_*((\xi_u)_{\mathbb{P}}) = \mathcal{O}_{\mathcal{U}} \boxtimes \Delta_*(\xi_u).$$

Now, proceed as in the proof of Lemma 4.8.   □

Lemma 4.10. *With notation as in (4) we have*

$$c_{u,v}^w(j) = \langle [\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}], \hat{p}^*[\mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j)] \cdot \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \rangle,$$

where the pairing

$$\langle \ , \ \rangle : K^0(\bar{Y}_{\mathbb{P}}) \otimes K_0(Y_{\mathbb{P}}) \to \mathbb{Z}$$

is defined (similar to (1)) by

$$\langle [\mathcal{S}], [\mathcal{F}] \rangle = \sum_i (-1)^i \chi(\bar{Y}_{\mathbb{P}}, \mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}}(\mathcal{S}, \mathcal{F})),$$

where χ denotes the Euler-Poincaré characteristic, and (as earlier) the map $\hat{p} : Y_{\mathbb{P}} \to \mathbb{P}$ denotes the projection.

Proof. First, we have

$$\langle [\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}], \hat{p}^*[\mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j)] \cdot \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \rangle = \langle \bar{p}^*[\mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j)] \cdot [\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}], \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \rangle,$$

where $\bar{p} : \bar{Y}_{\mathbb{P}} \to \mathbb{P}$ denotes the projection. To see this, first take a locally free finite resolution of $\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}$ on a quasi-compact open subset of $\bar{Y}_{\mathbb{P}}$ and a locally free finite resolution of $\mathcal{O}_{\mathbb{P}_j}(-\partial \mathbb{P}_j)$ on $\mathbb{P}$. Then, use the fact that for a locally free sheaf $\mathcal{F}$ on a quasi-compact open subset of $\bar{Y}_{\mathbb{P}}$ and a locally free sheaf $\mathcal{G}$ on $\mathbb{P}$, we have

$$\langle \mathcal{F}, \hat{p}^*(\mathcal{G}) \cdot \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \rangle = \langle \bar{p}^*(\mathcal{G}) \cdot \mathcal{F}, \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \rangle.$$

Now, the lemma follows from (4), Lemma 4.8, Proposition 3.10 and [Ku1, Identity 20].   □

We now introduce the mixing group Γ which acts on $\bar{Y}_{\mathbb{P}}$.

Definition 4.11 (Mixing group Γ). Let T act on B via

$$t \cdot b = tbt^{-1}$$

for $t \in T, b \in B$. This action induces a natural action of ΔT on $B \times B$. Consider the ind-group scheme over $\mathbb{P}$:

$$(B^2)_{\mathbb{P}} = E(T)_{\mathbb{P}} \times^T B^2 \to \mathbb{P}.$$

Let Γ_0 denote the group of global sections of $(B^2)_{\mathbb{P}}$ under pointwise multiplication. Since $GL(N+1)^r$ acts canonically on $(B^2)_{\mathbb{P}}$ in a way compatible with its action on $\mathbb{P}$, it also acts on Γ_0 via inverse pullback. We define the mixing group Γ to be the semidirect product $\Gamma := \Gamma_0 \rtimes GL(N+1)^r$:

$$1 \to \Gamma_0 \to \Gamma \to GL(N+1)^r \to 1.$$

By [Ku1, Lemmas 4.7 and 4.8 (more precisely, the paragraph following these lemmas)], we have the following two lemmas:

Lemma 4.12. Γ *is connected.*

Lemma 4.13. *For any $\bar{e} \in \mathbb{P}$ and any (b, b') in the fiber of $(B^2)_{\mathbb{P}}$ over $\bar{e}$ there exists a section $\gamma \in \Gamma_0$ such that $\gamma(\bar{e}) = (b, b')$.*

We define the action of Γ on $\bar{Y}_{\mathbb{P}}$ via

$$(\gamma, g) \cdot [e, (x, y)] = [ge, \gamma(ge) \cdot (x, y)]$$

for $\gamma \in \Gamma_0, g \in GL(N+1)^r, e \in E(T)_{\mathbb{P}}$, and $(x, y) \in \bar{Y}$, where the action of Γ_0 is via the standard action of $B \times B$ on $\bar{Y} = \bar{X}^2$. It follows from Lemma 4.13 that the orbits of the Γ-action on $\bar{Y}_{\mathbb{P}}$ are precisely equal to $\{(C_w \times C_{w'})_{\mathbb{P}}\}$.

5. STATEMENT OF MAIN RESULTS

We now come to our main technical result.

Theorem 5.1. *For general $\gamma \in \Gamma$ and any $u, v, w \in W, j \in [N]^r$ we have:*

a) *For all $i > 0$,*

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}, \hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right) = 0.$$

b) *Assume $c_{u,v}^w(j) \neq 0$. For all $p \neq |j| + \ell(w) - \ell(u) - \ell(v)$,*

$$H^p \left(\bar{Y}_{\mathbb{P}}, \gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right) \right) = 0,$$

where $|j| := \sum_{i=1}^r j_i$.

Proof. Deferred to the later sections. Part a) is proved in Section 6, while part b) is proved in Section 9. $\square$

Since Γ is connected, Lemmas 4.10, 4.9, and Theorem 5.1 together give:

Corollary 5.2. $(-1)^{\ell(w) - \ell(u) - \ell(v) + |j|} c_{u,v}^w(j) \in \mathbb{Z}_{\geq 0}$.

As an immediate consequence of Corollary 5.2 and Lemma 4.7 we get:

Theorem 5.3. *For any symmetrizable Kac-Moody group G and any $u, v, w \in W$, the structure constants of the product in the basis $\{[\mathcal{O}_{X^w}]\}$ in $K_T^0(\bar{X})$ satisfy*

$$(-1)^{\ell(w) + \ell(u) + \ell(v)} d_{u,v}^w \in \mathbb{Z}_{\geq 0}[(e^{-\alpha_1} - 1), \ldots, (e^{-\alpha_r} - 1)].$$

6. Proof of part a) of Theorem 5.1

The key tool used to prove part a) of Theorem 5.1 is the following transversality result, taken from [AGM, Theorem 2.3] (originally due to Sierra).

Theorem 6.1. *Let X be a variety with a left action of an algebraic group G and let $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\mathcal{F}$ is homologically transverse to the closures of the G-orbits on X. Then, for each coherent sheaf $\mathcal{E}$ on X, there is a Zariski-dense open set $U \subseteq G$ such that $\mathscr{T}or_j^{\mathcal{O}_X}(\mathcal{F}, g_*\mathcal{E}) = 0$ for all $j \geq 1$ and all $g \in U$.*

Theorem 5.1 part a) is a particular case of the following slightly more general result by taking $\mathcal{E} = \hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{Y_\mathbb{P}}} \tilde{\Delta}_*((\xi_w)_\mathbb{P})$.

Theorem 6.2. *Let $w \in W$ and let $\mathcal{E}$ be a coherent sheaf on $(Y_w)_\mathbb{P} := (X_w \times X_w)_\mathbb{P}$. Then, for general $\gamma \in \Gamma$ and any $u, v \in W$ we have:*

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_\mathbb{P}}}\left(\gamma_* \mathcal{O}_{(X^u \times X^v)_\mathbb{P}}, \mathcal{E}\right) = 0, \ \forall i > 0.$$

Proof. We first show that for any $\gamma \in \Gamma$,

$$(5) \qquad \mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_\mathbb{P}}}\left(\mathcal{O}_{(X^u \times X^v)_\mathbb{P}}, \gamma_*\mathcal{E}\right) = \mathscr{T}or_i^{\mathcal{O}_{(Y_w)_\mathbb{P}}}\left(\mathcal{O}_{(Y_w)_\mathbb{P}} \otimes_{\mathcal{O}_{\bar{Y}_\mathbb{P}}} \mathcal{O}_{(X^u \times X^v)_\mathbb{P}}, \gamma_*\mathcal{E}\right).$$

Since $\gamma_*\mathcal{E}$ is a coherent sheaf on $(Y_w)_\mathbb{P}$, we can replace $\gamma_*\mathcal{E}$ by $\mathcal{E}$ itself.

As the assertion is local on $\mathbb{P}$ we may assume $\bar{Y}_\mathbb{P} \simeq \mathbb{P} \times \bar{Y}$ and that

$$\mathcal{O}_{(X^u \times X^v)_\mathbb{P}} \simeq \mathcal{O}_\mathbb{P} \boxtimes (\mathcal{O}_{X^u} \boxtimes \mathcal{O}_{X^v}),$$

$$\mathcal{O}_{(Y_w)_\mathbb{P}} \simeq \mathcal{O}_\mathbb{P} \boxtimes (\mathcal{O}_{X_w} \boxtimes \mathcal{O}_{X_w}).$$

To simplify notation let $A = \mathcal{O}_{\bar{Y}_\mathbb{P}}$, $B = \mathcal{O}_{(Y_w)_\mathbb{P}}$, $M = \mathcal{O}_{(X^u \times X^v)_\mathbb{P}}$ and $N = \mathcal{E}$. Take an A-free resolution $\mathcal{F}_\bullet \to M$ on an open subset of $\bar{Y}_\mathbb{P}$ of the form $(V^{u'} \times V^{v'})_\mathbb{P}$, where V^u is defined as in Lemma 3.3. Then, the homology of the chain complex $B \otimes_A \mathcal{F}_\bullet$ is by definition $\mathscr{T}or_\bullet^A(M, B)$. Moreover,

$$\mathscr{T}or_i^A(M, B) = 0, \ \forall i > 0.$$

Indeed, locally on $\mathbb{P}$,

$$\mathscr{T}or_i^A(M, B) = \mathscr{T}or_i^{\mathcal{O}_\mathbb{P} \boxtimes \mathcal{O}_{\bar{Y}}}\left(\mathcal{O}_\mathbb{P} \boxtimes \mathcal{O}_{(X^u \times X^v)}, \mathcal{O}_\mathbb{P} \boxtimes \mathcal{O}_{(X_w \times X_w)}\right)$$

$$= \bigoplus_{j+k=i} \mathscr{T}or_j^{\mathcal{O}_\mathbb{P}}(\mathcal{O}_\mathbb{P}, \mathcal{O}_\mathbb{P}) \otimes \mathscr{T}or_k^{\mathcal{O}_{\bar{Y}}}(\mathcal{O}_{X^u \times X^v}, \mathcal{O}_{X_w \times X_w})$$

$$= 0, \ \forall i > 0$$

by Lemma 3.7 and the Kunneth formula. Hence, $B \otimes_A \mathcal{F}_\bullet$ is a B-free resolution of $B \otimes_A M$.

Thus, the homology of the chain complex $N \otimes_B (B \otimes_A \mathcal{F}_\bullet)$ is equal to $\mathscr{T}or_\bullet^B(B \otimes_A M, N)$; but,

$$N \otimes_B (B \otimes_A \mathcal{F}_\bullet) = (N \otimes_B B) \otimes_A \mathcal{F}_\bullet = N \otimes_A \mathcal{F}_\bullet,$$

so the homology is also equal to $\mathscr{T}or_\bullet^A(M, N)$. Hence,

$$\mathscr{T}or_\bullet^B(B \otimes_A M, N) = \mathscr{T}or_\bullet^A(M, N)$$

as desired. This proves (5).

Now, by Lemma 4.13, the closures of the Γ-orbits of $(Y_w)_\mathbb{P}$ are precisely $(X_x \times X_y)_\mathbb{P}$, where $x, y \leq w$. Equation (5) implies that the sheaf $\mathcal{F}$ defined by

$$\mathcal{F} := \mathcal{O}_{(Y_w)_\mathbb{P}} \otimes_{\mathcal{O}_{\bar{Y}_\mathbb{P}}} \mathcal{O}_{(X^u \times X^v)_\mathbb{P}}$$

is homologically transverse to the Γ-orbit closures in $(Y_w)_\mathbb{P}$. Indeed, since $\mathcal{O}_{(X_x \times X_y)_\mathbb{P}}$ is a coherent $\mathcal{O}_{(Y_w)_\mathbb{P}}$-module when $x, y \leq w$, equation (5) gives

$$\mathscr{T}or_i^{\mathcal{O}_{(Y_w)_\mathbb{P}}} \left(\mathcal{F}, \mathcal{O}_{(X_x \times X_y)_\mathbb{P}} \right) = \mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_\mathbb{P}}} \left(\mathcal{O}_{(X^u \times X^v)_\mathbb{P}}, \mathcal{O}_{(X_x \times X_y)_\mathbb{P}} \right)$$
$$= 0, \ \forall i > 0$$

by Lemma 3.7 and the Kunneth formula.

Thus, by Theorem 6.1 (with $G = \Gamma, X = (Y_w)_\mathbb{P}$, and $\mathcal{E}$ and $\mathcal{F}$ as above), we conclude that for general $\gamma \in \Gamma$,

(6) $\qquad \mathscr{T}or_i^{\mathcal{O}_{(Y_w)_\mathbb{P}}} \left(\mathcal{O}_{(Y_w)_\mathbb{P}} \otimes_{\mathcal{O}_{\bar{Y}_\mathbb{P}}} \mathcal{O}_{(X^u \times X^v)_\mathbb{P}}, \gamma_* \mathcal{E} \right) = 0, \ \forall i > 0.$

Here we note that although Γ is infinite dimensional, the action of Γ on $(Y_w)_\mathbb{P}$ factors through the action of a finite dimensional quotient group of Γ.

Now, (5) gives

$$\mathscr{T}or_i^{\mathcal{O}_{\bar{Y}_\mathbb{P}}} \left(\mathcal{O}_{(X^u \times X^v)_\mathbb{P}}, \gamma_* \mathcal{E} \right) = 0, \ \forall i > 0,$$

which is equivalent to the desired vanishing. $\qquad \square$

7. THE SCHEMES $\mathcal{Z}$ AND $\partial\mathcal{Z}$

For $u, v \leq w$ we use the notation $X_w^{u,v} := X_w^u \times X_w^v$. We also write $X_w^2 := X_w \times X_w$. For any $j \in [N]^r$, we let $(X_w)_j$ denote the inverse image of $\mathbb{P}_j$ through the map $E(T)_\mathbb{P} \times^T X_w \to \mathbb{P}$.

Similarly, for $u, v \leq w$ we write $Z_w^{u,v} := Z_w^u \times Z_w^v$, where Z_w^u is the T-equivariant desingularization of X_w^u as in [Ku1, Theorem 6.8]. We also write $Z_w^2 := Z_w \times Z_w$, where Z_w is a BSDH variety as in [Ku2, §7.1.3]. For any $j \in [N]^r$, we let $(Z_w)_j$ denote the inverse image of $\mathbb{P}_j$ through the map $E(T)_\mathbb{P} \times^T Z_w \to \mathbb{P}$.

The action of B on Z_w factors through the action of a finite dimensional quotient group $\bar{B}$ containing the maximal torus H. Further, the action of Γ on $(X_w^2)_\mathbb{P}$ descends to an action of the finite dimensional quotient group

$$\bar{\Gamma} := \bar{\Gamma}_0 \rtimes GL(N+1)^r,$$

where $\bar{\Gamma}_0$ is the group of global sections of the bundle $E(T)_\mathbb{P} \times^T (\bar{B}^2) \to \mathbb{P}$.

From [Ku1, Lemmas 6.11 and 6.12] we have

Lemma 7.1. Let $u, v \leq w$. The map

$$m : \bar{\Gamma} \times (X_w^{u,v})_\mathbb{P} \to (X_w^2)_\mathbb{P}, m(\gamma, x) = \gamma \cdot \pi_2(x),$$

is flat, where $\pi_2 : (X_w^u \times X_w^v)_\mathbb{P} \to (X_w^2)_\mathbb{P}$ is induced from the canonical map $X_w^u \times X_w^v \to X_w^2$.

Lemma 7.2. Let $u, v \leq w$. The map

$$\tilde{m} : \bar{\Gamma} \times (Z_w^{u,v})_\mathbb{P} \to (Z_w^2)_\mathbb{P}, \tilde{m}(\gamma, x) = \gamma \cdot \tilde{\pi}_2(x),$$

is smooth, where $\tilde{\pi}_2 : (Z_w^u \times Z_w^v)_\mathbb{P} \to (Z_w^2)_\mathbb{P}$ is induced from the canonical map $Z_w^u \times Z_w^v \to Z_w^2$.

Define $\mathcal{Z}$ to be the fiber product

$$\mathcal{Z} := \left(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\right) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((X_w)_j)$$

and $\widetilde{\mathcal{Z}}$ to be the fiber product

$$\widetilde{\mathcal{Z}} := \left(\bar{\Gamma} \times (Z_w^{u,v})_{\mathbb{P}}\right) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}((Z_w)_j)$$

as in the commutative diagram:

In the above diagram, $\square$ denotes a fiber square. Note that the maps θ and β above are desingularizations. The maps $\pi : \mathcal{Z} \to \bar{\Gamma}$ and $\tilde{\pi} : \widetilde{\mathcal{Z}} \to \bar{\Gamma}$ are induced by the projections onto the first factor. The map $f : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is defined by $f := \theta \circ \tilde{\iota}$. It is clear that the image of f is indeed contained inside of $\mathcal{Z}$ using commutativity of the diagram, along with the fact that $\beta(\tilde{\Delta}((Z_w)_j)) = \tilde{\Delta}((X_w)_j)$.

We define the boundary of $(X_w)_j$ by

$$\partial((X_w)_j) := (\partial X_w)_j \cup (X_w)_{\partial \mathbb{P}_j}$$

and similarly define the boundary of $(Z_w)_j$ by

$$\partial((Z_w)_j) := (\partial Z_w)_j \cup (Z_w)_{\partial \mathbb{P}_j},$$

where $\partial Z_w := \varphi^{-1}(\partial X_w)$ and $\varphi : Z_w \to X_w$ denotes the desingularization.

We define the boundary of $\mathcal{Z}$ by

$$\partial \mathcal{Z} := \left(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\right) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}(\partial((X_w)_j))$$

and similarly define the boundary of $\tilde{\mathcal{Z}}$ by

$$\partial \tilde{\mathcal{Z}} := \left(\bar{\Gamma} \times (Z_w^{u,v})_{\mathbb{P}}\right) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}(\partial((Z_w)_j)).$$

Observe that $f^{-1}(\partial \mathcal{Z}) = \partial \widetilde{\mathcal{Z}}$ is the scheme-theoretic inverse image.

We will need the following lemmas, which are restatements of [Ku1, Lemmas 7.2 and 7.3], respectively, ([Ku1, Lemma 7.3] is originally from [FP, Lemma on page 108]).

Lemma 7.3. *Let $f : W \to X$ be a flat morphism from a pure-dimensional* CM *scheme W of finite type over $\mathbb{C}$ to a* CM *irreducible variety X and let Y be a closed* CM *subscheme of X of pure codimension d. Set $Z := f^{-1}(Y)$. If $\mathrm{codim}_W(Z) \geq d$, then equality holds and Z is* CM.

Lemma 7.4. *Let $f : W \to X$ be a morphism from a pure-dimensional* CM *scheme W of finite type over $\mathbb{C}$ to a smooth irreducible variety X and let Y be a closed* CM *subscheme of X of pure codimension d. Set $Z := f^{-1}(Y)$. If $\mathrm{codim}_W(Z) \geq d$, then equality holds and Z is* CM.

Proposition 7.5. *The scheme $\mathcal{Z}$ is normal, irreducible, and* CM, *of dimension*

$$\dim \mathcal{Z} = |j| + \ell(w) - \ell(u) - \ell(v) + \dim \bar{\Gamma}.$$

The scheme $\widetilde{\mathcal{Z}}$ is irreducible, and the map $f : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is a proper birational map. Hence, $\widetilde{\mathcal{Z}}$ is a desingularization of $\mathcal{Z}$.

Proof. By [H, Chapter III, Corollary 9.6], the fibers of m are pure of dimension

$$\dim \bar{\Gamma} + \dim(X_w^{u,v})_{\mathbb{P}} - \dim(X_w^2)_{\mathbb{P}}.$$

Since the fibers of μ are the same as those of m, applying loc. cit. to μ gives that $\mathcal{Z}$ is pure of dimension

$$\dim \mathcal{Z} = \dim \bar{\Gamma} + \dim(X_w^{u,v})_{\mathbb{P}} - \dim(X_w^2)_{\mathbb{P}} + \dim \tilde{\Delta}((X_w)_j)$$

$$= |j| + \ell(w) - \ell(u) - \ell(v) + \dim \bar{\Gamma}.$$

The remainder of the proof of the proposition follows from the proof of [Ku1, Proposition 7.4 and Lemma 7.5] (Note that there is a slight difference in the definition of $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$ between ours and that in [Ku1].) Further, $\widetilde{\mathcal{Z}}$ is non-singular since $\tilde{\mu}$ is a smooth morphism with non-singular base. $\square$

Lemma 7.6. *Let G be a group acting on a set X and let $Y \subset X$. Consider the action map $m : G \times Y \to X$. For $x \in X$ denote the orbit of x by $O(x)$ and the stabilizer by $\mathrm{Stab}(x)$. Then, $\mathrm{Stab}(x)$ acts on the fiber $m^{-1}(x)$ and $\mathrm{Stab}(x) \backslash m^{-1}(x) \simeq O(x) \cap Y$.*

Proof. It is easy to check that

$$m^{-1}(x) = \left\{ (g, h^{-1}x) : h \in G, \ h^{-1}x \in Y, \ g \in \mathrm{Stab}(x) \cdot h \right\}.$$

Thus, $\mathrm{Stab}(x)$ acts on $m^{-1}(x)$ by left multiplication on the left component. Since every element of $O(x) \cap Y$ is of the form $h^{-1}x$ for some $h \in G$, the second projection $m^{-1}(x) \to O(x) \cap Y$ is surjective. This map clearly factors through the quotient to give a map $\mathrm{Stab}(x) \backslash m^{-1}(x) \to O(x) \cap Y$. To show this induced map is injective, note first that each class has a representative of the form $(h, h^{-1}x)$. Now, if $(h_1, h_1^{-1}x)$ and $(h_2, h_2^{-1}x)$ satisfy $h_1^{-1}x = h_2^{-1}x$, then $h_2 h_1^{-1}x = x$, i.e., $h_2 h_1^{-1} \in \mathrm{Stab}(x)$, i.e., $h_2 \in \mathrm{Stab}(x) \cdot h_1$, i.e., $(h_1, h_1^{-1}x)$ and $(h_2, h_2^{-1}x)$ belong to the same class. $\square$

Proposition 7.7. *The scheme $\mathcal{Z}$ has rational singularities.*

Proof. Since μ is flat and $\tilde{\Delta}((X_w)_j)$ has rational singularities, by [El, Théorème 5] it is sufficient to show that the fibers of μ are disjoint unions of irreducible varieties with rational singularities (in fact, the fibers of μ are irreducible, but we do not provide a proof here as we do not need this fact).

Let $x \in \tilde{\Delta}((C_{w'})_j)$, where $w' \leq w$. Then, by Lemmas 7.6 and 4.13, we have $\mathrm{Stab}(x) \backslash \mu^{-1}(x) \simeq (X^u \cap C_{w'} \times X^v \cap C_{w'})_{\mathbb{P}}$, where $\mathrm{Stab}(x)$ is taken with respect to the action of $\bar{\Gamma}$ on $(X_w^2)_{\mathbb{P}}$. By [Se, Proposition 3, §2.5], the quotient map $\bar{\Gamma} \to \mathrm{Stab}(x) \backslash \bar{\Gamma}$ is locally trivial in the étale topology.

Consider the pullback diagram:

$$
\begin{array}{ccc}
\mu^{-1}(x) & \subseteq & \bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}} \\
\big\downarrow & & \big\downarrow \\
\mathrm{Stab}(x) \backslash \mu^{-1}(x) & \subseteq & \big(\mathrm{Stab}(x) \backslash \bar{\Gamma}\big) \times (X_w^{u,v})_{\mathbb{P}}
\end{array}
$$

Since the right vertical map is a locally trivial fibration in the étale topology, the left vertical map is too. Now, $\mathrm{Stab}(x) \backslash \mu^{-1}(x) \simeq (X^u \cap C_{w'} \times X^v \cap C_{w'})_{\mathbb{P}}$ has rational singularities by [KuS, Theorem 3.1]. Further, $\mathrm{Stab}(x)$ being smooth and $\mu^{-1}(x) \to \mathrm{Stab}(x) \backslash \mu^{-1}(x)$ being locally trivial in the étale topology, we get that $\mu^{-1}(x)$ is a disjoint union of irreducible varieties with rational singularities by [KM, Corollary 5.11]. $\square$

Proposition 7.8. *The scheme $\partial \mathcal{Z}$ is pure of codimension 1 in $\mathcal{Z}$ and is* CM.

Proof. Using the fact that $\tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ and $\tilde{\Delta}((\partial X_w)_j)$ are equidimensional, applying [H, Chapter III, Corollary 9.6] to their irreducible components gives that $\big(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\big) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ and $\big(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\big) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((\partial X_w)_j)$ are both pure of dimension

$$
\dim \bar{\Gamma} + \dim(X_w^{u,v})_{\mathbb{P}} - \dim(X_w^2)_{\mathbb{P}} + \dim \tilde{\Delta}((X_w)_j) - 1.
$$

Hence, $\partial \mathcal{Z}$ is pure of codimension 1 in $\mathcal{Z}$.

A similar argument also gives that $\big(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\big) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})$ is pure of dimension

$$
\dim \bar{\Gamma} + \dim(X_w^{u,v})_{\mathbb{P}} - \dim(X_w^2)_{\mathbb{P}} + \dim \tilde{\Delta}((X_w)_j) - 2.
$$

Next, we show that $\partial \mathcal{Z}$ is CM. Since $(X_w)_j$ is a locally trivial fibration over $\mathbb{P}_j$, it is locally a product of CM schemes and hence is CM. Also, since $\partial \mathbb{P}_j$ is CM, we similarly have that $(X_w)_{\partial \mathbb{P}_j}$ and hence $\tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ is CM. Now, applying Lemma 7.3 to $\mu : \mathcal{Z} \to \tilde{\Delta}((X_w)_j)$ gives that $\big(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\big) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ is CM, since $\tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ and $\mu^{-1}(\tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})) = \big(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\big) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ are of pure codimension 1 in $\tilde{\Delta}((X_w)_j)$ and $\mathcal{Z}$, respectively.

Observe that ∂X_w is CM. To prove this, use Proposition 3.6 and the argument as in the proof of [Ku1, Corollary 10.5] by taking an embedding of X_w into a smooth projective variety. Thus, $(\partial X_w)_j$ is locally a product of CM schemes and hence is CM, and hence so is $\tilde{\Delta}((\partial X_w)_j)$. Now, Lemma 7.3 applied to $\mu : \mathcal{Z} \to \tilde{\Delta}((X_w)_j)$ gives that

$$
\big(\bar{\Gamma} \times (X_w^{u,v})_{\mathbb{P}}\big) \times_{(X_w^2)_{\mathbb{P}}} \tilde{\Delta}((\partial X_w)_j)
$$

is CM since $\tilde{\Delta}((\partial X_w)_j)$ and $\mu^{-1}(\tilde{\Delta}((\partial X_w)_j)) = \left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_j)$ are of pure codimension 1 in $\tilde{\Delta}((X_w)_j)$ and $\mathcal{Z}$, respectively.

The intersection

$$\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})$$

$$= \left(\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})\right) \cap \left(\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_j)\right)$$

is of pure codimension 1 in both $\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((X_w)_{\partial \mathbb{P}_j})$ and $\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_j)$. Now, [E, Exercise 18.13] gives that the union $\partial \mathcal{Z}$ is CM iff the intersection $\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})$ is. But Lemma 7.3 applied to $\mu : \mathcal{Z} \to \tilde{\Delta}((X_w)_j)$ gives that $\left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})$ is CM, since $\tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})$ and $\mu^{-1}(\tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})) = \left(\bar{\Gamma} \times (X_w^{u,v})_\mathbb{P}\right) \times_{(X_w^2)_\mathbb{P}} \tilde{\Delta}((\partial X_w)_{\partial \mathbb{P}_j})$ are of pure codimension 2 in $\tilde{\Delta}((X_w)_j)$ and $\mathcal{Z}$, respectively. $\square$

Lemma 7.9. *The morphism $\mu : \mathcal{Z} \to \tilde{\Delta}((X_w)_j)$ is affine.*

Proof. Since $\tilde{\Delta}((X_w)_j)$ is a closed subscheme of $(X_w^2)_\mathbb{P}$ it suffices to show that the map $m : \bar{\Gamma} \times (X_w^{u,v})_\mathbb{P} \to (X_w^2)_\mathbb{P}$ is an affine morphism. Now, if $f : X \to Y$ is an affine morphism and $Z \subset X$ is a closed subscheme, then $f|_Z : Z \to Y$ is clearly an affine morphism. Thus, it suffices to show that $\bar{\Gamma} \times (X_w^2)_\mathbb{P} \to (X_w^2)_\mathbb{P}$ is an affine morphism. Further, if X and Y are total spaces of principal T-bundles and $f : X \to Y$ is a T-equivariant map, then f is affine iff $\bar{f} : X/T \to Y/T$ is affine. Thus, it suffices to show that

$$\hat{\mu} : \bar{\Gamma} \times (E(T)_\mathbb{P} \times X_w^2) \to E(T)_\mathbb{P} \times X_w^2$$

is affine. Recall that $\bar{\Gamma} = \bar{\Gamma}_0 \rtimes GL(N+1)^r$ and $\hat{\mu}$ is given by $\hat{\mu}((\sigma, g), (e, x)) = (ge, \sigma(ge) \cdot x)$, where $\sigma \in \bar{\Gamma}_0, g \in GL(N+1)^r, e \in E(T)_\mathbb{P}, x \in X_w^2$. Write $\hat{\mu}$ as a composite $\hat{\mu} = \mu_3 \circ \mu_2 \circ \mu_1$, where

$$\mu_1 : \bar{\Gamma}_0 \times GL(N+1)^r \times E(T)_\mathbb{P} \times X_w^2 \to \bar{\Gamma}_0 \times E(T)_\mathbb{P} \times X_w^2, \ (\sigma, g, e, x) \mapsto (\sigma, g \cdot e, x),$$

$$\mu_2 : \bar{\Gamma}_0 \times E(T)_\mathbb{P} \times X_w^2 \to \bar{B}^2 \times E(T)_\mathbb{P} \times X_w^2, \ (\sigma, e, x) \mapsto (\sigma(e), e, x),$$

$$\mu_3 : \bar{B}^2 \times E(T)_\mathbb{P} \times X_w^2 \to E(T)_\mathbb{P} \times X_w^2, \ ((b_1, b_2), e, (x, y)) \mapsto (e, b_1 x, b_2 y).$$

As the composition of two affine morphisms is affine, it suffices to prove that μ_1, μ_2, μ_3 are affine. Moreover, if $f : X \to Y$ is affine, then $f \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is affine for any scheme Z. Hence, it suffices to prove that the following maps $\hat{\mu}_1, \hat{\mu}_2$, and $\hat{\mu}_3$ are affine:

$$\hat{\mu}_1 : GL(N+1)^r \times E(T)_\mathbb{P} \to E(T)_\mathbb{P}, \ (g, e) \mapsto g \cdot e,$$

$$\hat{\mu}_2 : \bar{\Gamma}_0 \times E(T)_\mathbb{P} \to \bar{B}^2 \times E(T)_\mathbb{P}, \ (\sigma, e) \mapsto (\sigma(e), e),$$

$$\hat{\mu}_3 : \bar{B}^2 \times X_w^2 \to X_w^2, \ ((b_1, b_2), (x, y)) \mapsto (b_1 x, b_2 y).$$

(1) $\hat{\mu}_1$ **is affine**: Since $E(T)_\mathbb{P} = (\mathbb{C}^{N+1} \setminus \{0\})^r$, it suffices to prove that $\theta : GL(N+1) \times (\mathbb{C}^{N+1} \setminus \{0\}) \to \mathbb{C}^{N+1} \setminus \{0\}, \ (g, v) \mapsto g \cdot v$ is affine. Now, consider the map $\bar{\theta} : GL(N+1) \times \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}, \ (g, v) \mapsto g \cdot v$. Since both the domain and codomain are affine, $\bar{\theta}$ is an affine morphism. Moreover, $\theta = \bar{\theta}|_{\bar{\theta}^{-1}(\mathbb{C}^{N+1} \setminus \{0\})}$. Thus, θ is affine.

(2) $\hat{\mu}_2$ **is affine**: Take an affine open subset $\mathcal{U} \subset E(T)_\mathbb{P}$. Then, $\bar{B}^2 \times \mathcal{U}$ is an affine open subset in $\bar{B}^2 \times E(T)_\mathbb{P}$. Now, $\hat{\mu}_2^{-1}(\bar{B}^2 \times \mathcal{U}) = \bar{\Gamma}_0 \times \mathcal{U}$. Since $\bar{\Gamma}_0$ is affine, so is $\bar{\Gamma}_0 \times \mathcal{U}$. Thus, $\hat{\mu}_2$ is affine.

(3) $\hat{\mu}_3$ **is affine**: It suffices to prove that $\delta : \bar{B} \times X_w \to X_w$, $(b, x) \mapsto bx$ is affine. Take an affine open subset $V \subset X_w$. Then, $\delta^{-1}(V) = \bigcup_{b \in \bar{B}}(b, b^{-1}V)$. Consider the scheme isomorphism $i : \bar{B} \times X_w \mapsto \bar{B} \times X_w$, $(b, x) \mapsto (b, b \cdot x)$. Then, $i(\delta^{-1}(V)) = \bar{B} \times V$. But, since $\bar{B} \times V$ is affine, so is $\delta^{-1}(V)$. Thus, δ is an affine morphism and hence so is $\hat{\mu}_3$. $\square$

Let $\pi : \mathcal{Z} \to \bar{\Gamma}$ denote the projection onto the first factor and $\pi_1 : \partial\mathcal{Z} \to \bar{\Gamma}$ denote the restriction of π to $\partial\mathcal{Z}$. We define the fibers

$$(7) \qquad N_\gamma := \pi^{-1}(\gamma) \simeq \gamma((X_w^{u,v})_\mathbb{P}) \cap \tilde{\Delta}((X_w)_j)$$

and

$$(8) \qquad M_\gamma := \pi_1^{-1}(\gamma) \simeq \gamma((X_w^{u,v})_\mathbb{P}) \cap \tilde{\Delta}(\partial((X_w)_j)).$$

Corollary 7.10. *Assume that $c_{u,v}^w(j) \neq 0$, where $c_{u,v}^w(j)$ are defined by (4). Then, for general $\gamma \in \bar{\Gamma}$, we have that N_γ (defined by (7)) is CM of pure dimension. In fact, for $\gamma \in \bar{\Gamma}$ such that*

$$\dim N_\gamma = \dim \mathcal{Z} - \dim \bar{\Gamma} = |j| + \ell(w) - \ell(u) - \ell(v),$$

N_γ is CM, and this condition is satisfied for general γ.

Similarly, if $|j| + \ell(w) - \ell(u) - \ell(v) > 0$, for general $\gamma \in \bar{\Gamma}$, we have that M_γ (defined by 8) is CM of pure codimension 1 in N_γ. If $|j| + \ell(w) - \ell(u) - \ell(v) = 0$, for general $\gamma \in \bar{\Gamma}$, we have that M_γ is empty.

Proof. First we show that π is surjective when $c_{u,v}^w(j) \neq 0$. From the definition of π we have that

$$\mathrm{Im}\, \pi = \left\{ \gamma \in \bar{\Gamma} : \gamma((X_w^{u,v})_\mathbb{P}) \cap \tilde{\Delta}((X_w)_j) \neq \emptyset \right\}.$$

Since $\bar{\Gamma}$ is connected, Lemma 4.10 along with the assumption $c_{u,v}^w(j) \neq 0$ gives that $\gamma((X^{u,v})_\mathbb{P}) \cap \tilde{\Delta}((X_w)_j) \neq \emptyset$ for any $\gamma \in \bar{\Gamma}$. Since $\gamma((X^{u,v})_\mathbb{P}) \cap \tilde{\Delta}((X_w)_j) = \gamma((X_w^{u,v})_\mathbb{P}) \cap \tilde{\Delta}((X_w)_j)$, we get that π is surjective.

Now, since $\mathcal{Z}$ is CM, applying Lemma 7.4 to π gives that if

$$(9) \qquad \mathrm{codim}_\mathcal{Z}(N_\gamma) = \dim \bar{\Gamma},$$

then N_γ is CM. By [S, Chapter I, §6.3, Theorem 1.25], this condition holds for γ in a dense open subset of $\bar{\Gamma}$. Thus, N_γ is CM for γ satisfying $\dim N_\gamma = \dim \mathcal{Z} - \dim \bar{\Gamma}$, and this condition is satisfied for general γ.

Next, we show that if $|j| + \ell(w) - \ell(u) - \ell(v) > 0$, then π_1 is surjective. First note that since π_1 is projective, if it is not surjective, its image is a proper closed subset of $\bar{\Gamma}$. Thus, for general $\gamma \in \bar{\Gamma}$, $M_\gamma = \emptyset$, i.e., $N_\gamma \subset \mathcal{Z} \setminus \partial\mathcal{Z}$. As μ is an affine morphism by Lemma 7.9, $\mathcal{Z} \setminus \partial\mathcal{Z}$ is affine. But, N_γ is projective of positive dimension (since $|j| + \ell(w) - \ell(u) - \ell(v) > 0$) which gives a contradiction.

In particular, there is at least one irreducible component of $\partial\mathcal{Z}$ on which π_1 is surjective. The other irreducible components are mapped to closed subsets of $\bar{\Gamma}$. Let $\mathcal{U}$ be the complement of the union of the images of the irreducible components on which π_1 is not surjective. Then, $\mathcal{U}$ is open, since there are only finitely many irreducible components. Applying [S, Chapter I, §6.3, Theorem 1.25] to π_1 on each irreducible component of $\partial\mathcal{Z}$ which surjects onto $\bar{\Gamma}$ and then intersecting with $\mathcal{U}$ gives that, for general $\gamma \in \bar{\Gamma}$,

$$(10) \qquad \mathrm{codim}_{\partial\mathcal{Z}}(M_\gamma) = \dim \bar{\Gamma}.$$

Thus, by Lemma 7.4, M_γ is CM. Moreover, (9) and (10) together imply that M_γ is pure of codimension 1 in N_γ.

In the case where $|j| + \ell(w) - \ell(u) - \ell(v) = 0$, we have $\dim \mathcal{Z} = \dim \bar{\Gamma}$, so that $\dim \partial \mathcal{Z} < \dim \bar{\Gamma}$. Thus, $\mathrm{Im}\, \pi_1$ is a proper closed subset of $\bar{\Gamma}$, and hence M_γ is empty for general γ. $\qquad\square$

Corollary 7.11. *Assume that $c_{u,v}^w(j) \neq 0$. Then, for general $\gamma \in \bar{\Gamma}$, we have*

$$\mathscr{E}xt_{\mathcal{O}_{N_\gamma}}^i(\mathcal{O}_{N_\gamma}(-M_\gamma), \omega_{N_\gamma}) = 0, \ \forall i > 0.$$

Proof. By [I, Proposition 11.33 and Corollary 11.43] we have

$$\mathscr{E}xt_{\mathcal{O}_{N_\gamma}}^i(\mathcal{O}_{M_\gamma}, \omega_{N_\gamma}) = 0, \ \text{unless } i = 1,$$

and, of course,

$$\mathscr{E}xt_{\mathcal{O}_{N_\gamma}}^i(\mathcal{O}_{N_\gamma}, \omega_{N_\gamma}) = 0, \ \forall i > 0.$$

Hence, the desired result follows from the long exact $\mathscr{E}xt$ sequence associated to the short exact sequence

$$0 \to \mathcal{O}_{N_\gamma}(-M_\gamma) \to \mathcal{O}_{N_\gamma} \to \mathcal{O}_{M_\gamma} \to 0. \qquad\square$$

8. Application of Kawamata-Viehweg vanishing

In this section we assume that $c_{u,v}^w(j) \neq 0$.

Let $f : X \to Y$ be a proper morphism between schemes, with X irreducible, and let $\mathcal{M}$ be a line bundle on X. Then, $\mathcal{M}$ is said to be f-*nef* if it has non-negative intersection with every curve contained in a fiber of f. The line bundle $\mathcal{M}$ is said to be f-*big* if rank $f_*(\mathcal{M}^k) > c \cdot k^n$ for some $c > 0$ and $k \gg 1$, where n is the dimension of a general fiber of f. A Weil divisor $D \subset X$ has *normal crossings* if all of its irreducible components intersect transversely. It is easy to see that ∂Z_w and hence $\partial((Z_w)_j)$ are normal crossings divisors in Z_w and $(Z_w)_j$, respectively.

We will need the following relative Kawamata-Viehweg vanishing theorem from [AGM, Theorem 2.4], which was originally extracted from [EV, Corollary 6.11].

Theorem 8.1. *Let $f : \widetilde{Z} \to Z$ be a proper surjective morphism of varieties, with $\widetilde{Z}$ non-singular. Let $\mathcal{M}$ be a line bundle on $\widetilde{Z}$ such that $\mathcal{M}^{\overline{N}}(-D)$ is f-nef and f-big for a normal crossing divisor $D = \sum_{j=1}^r a_j D_j$, where $0 < a_j < \overline{N}$ for all j. Then,*

$$R^i f_*(\mathcal{M} \otimes \omega_{\widetilde{Z}}) = 0, \ \forall i > 0.$$

Definition 8.2. We define the sheaf

$$\omega_{\mathcal{Z}}(\partial \mathcal{Z}) := \mathscr{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \omega_{\mathcal{Z}}).$$

Theorem 8.3. *We have $f_* \omega_{\widetilde{\mathcal{Z}}}(\partial \widetilde{\mathcal{Z}}) = \omega_{\mathcal{Z}}(\partial \mathcal{Z})$, where $\widetilde{\mathcal{Z}}, \mathcal{Z}, \partial \widetilde{\mathcal{Z}}, \partial \mathcal{Z}$, and the morphism $f : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ are defined after Lemma 7.2. (Observe that since $\mathcal{Z}$ is CM, the dualizing sheaf $\omega_{\mathcal{Z}}$ makes sense.)*

Proof. First, we claim

$$(11) \qquad \mathcal{O}_{\widetilde{\mathcal{Z}}}(\partial \widetilde{\mathcal{Z}}) \simeq \mathscr{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}}}(f^* \mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \mathcal{O}_{\widetilde{\mathcal{Z}}}),$$

where $\mathcal{O}_{\widetilde{\mathcal{Z}}}(\partial \widetilde{\mathcal{Z}}) := \mathscr{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}}}(\mathcal{O}_{\widetilde{\mathcal{Z}}}(-\partial \widetilde{\mathcal{Z}}), \mathcal{O}_{\widetilde{\mathcal{Z}}})$. To see this, first note that since $f^{-1}(\partial \mathcal{Z}) = \partial \widetilde{\mathcal{Z}}$ is the scheme-theoretic inverse image (cf. §6), the natural morphism $f^* \mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}) \to \mathcal{O}_{\widetilde{\mathcal{Z}}}(-\partial \widetilde{\mathcal{Z}})$ is surjective (cf., e.g., [Stacks, Tag 01HJ, Lemma 25.4.7]).

As f is a desingularization (cf. Proposition 7.5), the kernel of this morphism is supported on a proper closed subset of $\widetilde{\mathcal{Z}}$ and hence is a torsion sheaf. This implies that the dual map $\mathcal{O}_{\widetilde{\mathcal{Z}}}(\partial\widetilde{\mathcal{Z}}) \to \mathscr{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}}}(f^*\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \mathcal{O}_{\widetilde{\mathcal{Z}}})$ is an isomorphism, proving (11).

To complete the proof of the theorem, we compute:

$$f_*(\omega_{\widetilde{\mathcal{Z}}} \otimes \mathcal{O}_{\widetilde{\mathcal{Z}}}(\partial\widetilde{\mathcal{Z}})) = f_*(\omega_{\widetilde{\mathcal{Z}}} \otimes \mathscr{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}}}(f^*\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \mathcal{O}_{\widetilde{\mathcal{Z}}})), \text{ by (11)}$$

$$= f_*\mathscr{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}}}(f^*\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\widetilde{\mathcal{Z}}})$$

$$= \mathscr{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), f_*\omega_{\widetilde{\mathcal{Z}}}), \text{ by adjunction (cf. [H, Chapter II, §5])}$$

$$= \mathscr{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial\mathcal{Z}), \omega_{\mathcal{Z}}), \text{ by Proposition 7.7 and [KM, Theorem 5.10]}$$

$$= \omega_{\mathcal{Z}}(\partial\mathcal{Z}). \qquad \square$$

Lemma 8.4. *The homogeneous line bundle $\mathcal{L}(\rho)|_{X_w}$ (cf. §2) has a section with zero set exactly equal to ∂X_w.*

Proof. Consider the Borel-Weil homomorphism (cf. [Ku2, §8.1.21]) $\chi_w : L(\rho)^* \to H^0(X_w, \mathcal{L}(\rho))$ which is given by $\chi_w(f)(gB) = [g, f(ge_\rho)]$, where e_ρ is the highest weight vector of the irreducible highest weight $G^{\min}$-module $L(\rho)$ with highest weight ρ.

Consider the section $\chi_w(e_{w\rho}^*)$, where $e_{w\rho}$ is the weight vector of $L(\rho)$ with weight $w\rho$ and $e_{w\rho}^* \in L(\rho)^*$ is the linear form which takes the value 1 on $e_{w\rho}$ and 0 on any weight vector of $L(\rho)$ of weight different from $w\rho$. Let $w' \leq w$ and $b \in B$. We have

$$\chi_w(e_{w\rho}^*)(bw'B) = [bw', e_{w\rho}^*(bw'e_\rho)].$$

Now, for $w' < w$, we have $w'\rho > w\rho$ by [Ku2, Lemma 8.3.3]. Thus, $e_{w\rho}^*(bw'e_\rho) = 0$ for any $b \in B$ and $w' < w$. For $w' = w$, we have $e_{w\rho}^*(bwe_\rho) \neq 0$ for any $b \in B$. Hence, $\chi_w(e_{w\rho}^*)$ has zero set precisely equal to ∂X_w. $\qquad \square$

Lemma 8.5. *There is an ample line bundle $\mathcal{L}$ on $(X_w)_j$ with a section with zero set exactly equal to $\partial((X_w)_j)$.*

Proof. By the previous lemma, the ample line bundle $\mathcal{L}(\rho)|_{X_w}$ has a section with zero set exactly equal to ∂X_w. The T-equivariant line bundle $(e^{w\rho}\mathcal{L}(\rho))|_{X_w}$ gives rise to the line bundle

$$(e^{w\rho}\mathcal{L}(\rho)|_{X_w})_j := E(T)_{\mathbb{P}_j} \times^T ((e^{w\rho}\mathcal{L}(\rho))|_{X_w})$$

on $(X_w)_j$. Then, the section θ defined by $[e, x] \mapsto [e, 1_{w\rho} \otimes \chi_w(e_{w\rho}^*)x]$ for $e \in E(T)_{\mathbb{P}_j}$ and $x \in X_w$ has zero set exactly equal to $(\partial X_w)_j$, where $1_{w\rho}$ is a non-zero element of the line bundle $e^{w\rho}$ (cf. §2).

Now, let $\mathcal{H}$ be an ample line bundle on $\mathbb{P}_j$ with a section σ with zero set exactly $\partial\mathbb{P}_j$ and consider the bundle

$$\mathcal{L} := (e^{w\rho}\mathcal{L}(\rho)|_{X_w})_j \otimes p^*(\mathcal{H}^{\overline{N}}),$$

where $p : (X_w)_j \to \mathbb{P}_j$ is the projection. Let $\hat{\sigma}$ represent the pullback of σ. Then, the section $\theta \otimes (\hat{\sigma})^{\overline{N}}$ has zero set exactly equal to $\partial((X_w)_j)$. Furthermore, by [KM, Proposition 1.45], if $\overline{N}$ is large enough, $\mathcal{L}$ is ample. $\qquad\square$

Lemma 8.6. *Let $f : \widetilde{A} \to A$ be a proper birational map between normal, irreducible varieties and let $\pi : A \to B$ be a surjective proper morphism to a variety B. Let $\tilde{\mathcal{L}}$ be a π-big line bundle on A. Then, the pullback line bundle $f^* \tilde{\mathcal{L}}$ is $\tilde{\pi}$-big, where $\tilde{\pi} = \pi \circ f : \widetilde{A} \to B$.*

Proof. It suffices to show that rank $\tilde{\pi}_*(f^* \tilde{\mathcal{L}}^k) > c \cdot k^n$ for some $c > 0$ and $k \gg 1$, where n is the dimension of a general fiber of $\tilde{\pi}$.

By [S, Chapter 1, §6.3, Theorem 1.25], the dimension of a general fiber of π and the dimension of a general fiber of $\tilde{\pi}$ are both equal to $\dim A - \dim B$.

Now, $\tilde{\pi}_*(f^* \tilde{\mathcal{L}}^k) = \pi_* f_*(f^* \tilde{\mathcal{L}}^k) \simeq \pi_*(\tilde{\mathcal{L}}^k)$ by the projection formula, since A is normal. Hence, rank $\tilde{\pi}_*(f^* \tilde{\mathcal{L}}^k) > c \cdot k^n$ for some $c > 0$ and $k \gg 1$, where n is the dimension of a general fiber of π, since $\tilde{\mathcal{L}}$ is π-big. This proves the lemma. $\qquad\square$

Theorem 8.7. *Assume that $c_{u,v}^w(j) \neq 0$. Then, for all $i > 0$, we have $R^i \tilde{\pi}_* \omega_{\widetilde{\mathcal{Z}}}(\partial \widetilde{\mathcal{Z}}) = 0$ and $R^i f_* \omega_{\widetilde{\mathcal{Z}}}(\partial \widetilde{\mathcal{Z}}) = 0$, where $\tilde{\pi} : \widetilde{\mathcal{Z}} \to \bar{\Gamma}$ is the projection onto the first factor and $f : \widetilde{\mathcal{Z}} \to \mathcal{Z}$ is defined after Lemma 7.2.*

Proof. Since f is proper and birational by Proposition 7.5, it is surjective. By the proof of Corollary 7.10, the projection $\pi : \mathcal{Z} \to \bar{\Gamma}$ is also surjective. Thus, $\tilde{\pi} = \pi \circ f$ is also surjective. Moreover, $\tilde{\pi}$ is proper since it is the restriction of the projection $\bar{\Gamma} \times (Z_w^{u,v})_{\mathbb{P}} \to \bar{\Gamma}$ to the closed subset $\widetilde{\mathcal{Z}}$, where $(Z_w^{u,v})_{\mathbb{P}}$ is projective. Let $\mathcal{M}$ denote the line bundle $\mathcal{M} := \mathcal{O}_{\widetilde{\mathcal{Z}}}(\partial \widetilde{\mathcal{Z}})$. (Observe that $\partial \widetilde{\mathcal{Z}}$ is a divisor in non-singular $\widetilde{\mathcal{Z}}$.) By Theorem 8.1, it suffices to find a normal crossings divisor $D \subset \widetilde{\mathcal{Z}}$ such that the line bundle $\mathcal{M}^{\overline{N}}(-D)$ is $\tilde{\pi}$-nef, f-nef, $\tilde{\pi}$-big, and f-big and $D = \sum_{j=1}^r a_j D_j$, with $0 < a_j < \overline{N}$.

By Lemma 8.5 we may choose an ample line bundle $\mathcal{L}$ on $\tilde{\Delta}((X_w)_j)$ with a section with zero set exactly equal to $\tilde{\Delta}(\partial((X_w)_j))$. Let $\varphi : \tilde{\Delta}((Z_w)_j) \to \tilde{\Delta}((X_w)_j)$ denote the desingularization. Since $\partial((Z_w)_j)$ is a normal crossings divisor in $(Z_w)_j$, it follows that $\partial \widetilde{\mathcal{Z}} = \left(\bar{\Gamma} \times (Z_w^{u,v})_{\mathbb{P}} \right) \times_{(Z_w^2)_{\mathbb{P}}} \tilde{\Delta}(\partial((Z_w)_j))$ is a normal crossings divisor in $\widetilde{\mathcal{Z}}$. Write

$$\partial \widetilde{\mathcal{Z}} = D_1 + \cdots + D_\ell,$$

where D_i are the irreducible components. Since $\tilde{\mu}^* \varphi^* \mathcal{L}$ has a section with zero set precisely equal to $\partial \widetilde{\mathcal{Z}}$, it follows that

$$\tilde{\mu}^* \varphi^* \mathcal{L} = \mathcal{O}_{\widetilde{\mathcal{Z}}}(b_1 D_1 + \cdots + b_\ell D_\ell)$$

for some positive integers $b_1, \ldots, b_\ell$.

Let D be the divisor $D := a_1 D_1 + \cdots + a_\ell D_\ell$ where $a_i := \overline{N} - b_i$ for some integer $\overline{N}$ greater than all the b_i's. Since $\partial \widetilde{\mathcal{Z}}$ has normal crossings, so does D. Then,

$$\mathcal{M}^{\overline{N}}(-D) = \mathcal{O}_{\widetilde{\mathcal{Z}}}(b_1 D_1 + \cdots + b_\ell D_\ell) = \tilde{\mu}^* \varphi^* \mathcal{L}.$$

Since the fibers of $\tilde{\pi}$ are projective schemes and $\mathcal{L}$ is an ample line bundle on $\tilde{\Delta}((X_w)_j)$, the pullback $\tilde{\mu}^* \varphi^* \mathcal{L}$ restricted to the fibers of $\tilde{\pi}$ is nef, since the pullback of any ample line bundle under a morphism between projective varieties is nef

(cf. [D, Theorem 1.26, §1.9 and §1.29]). Thus, $\mathcal{M}^{\overline{N}}(-D)$ is $\tilde{\pi}$-nef. Since the fibers of f are contained in the fibers of $\tilde{\pi}$, $\mathcal{M}^{\overline{N}}(-D)$ is also f-nef.

Now, $\mathcal{M}^{\overline{N}}(-D)$ is f-big since f is birational by Proposition 7.5. It remains to show $\tilde{\pi}$-bigness. Clearly, $\mu : \mathcal{Z} \to \tilde{\Delta}((X_w)_j)$ is a closed embedding restricted to any fiber of the morphism $\pi : \mathcal{Z} \to \bar{\Gamma}$. Hence, the ample line bundle $\mathcal{L}$ on $\tilde{\Delta}((X_w)_j)$ pulls back to a π-big line bundle $\mu^*\mathcal{L}$ on $\mathcal{Z}$. Now, $\tilde{\pi}$-bigness of $\mathcal{M}^{\overline{N}}(-D) = \tilde{\mu}^*\varphi^*\mathcal{L} = f^*\mu^*\mathcal{L}$ follows from Lemma 8.6 and Proposition 7.5. □

Theorems 8.3 and 8.7 together with the Grothendieck spectral sequence give

Corollary 8.8. *For all $i > 0$ we have $R^i\pi_*\omega_{\mathcal{Z}}(\partial\mathcal{Z}) = 0$.*

9. Proof of part b) of Theorem 5.1

Recall the definitions of N_γ and M_γ from equations (7) and (8).
We also define
$$M_\gamma^1 := \gamma((X_w^{u,v})_{\mathbb{P}}) \cap \tilde{\Delta}((X_w)_{\partial\mathbb{P}_j})$$
and
$$M_\gamma^2 := \gamma((X_w^{u,v})_{\mathbb{P}}) \cap \tilde{\Delta}((\partial X_w)_j),$$
so that
$$M_\gamma = M_\gamma^1 \cup M_\gamma^2.$$

Lemma 9.1. *For general $\gamma \in \bar{\Gamma}$, the sheaf*
$$\gamma_*\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right)$$
is supported on N_γ and is equal to the sheaf $\mathcal{O}_{N_\gamma}(-M_\gamma)$, where $\hat{p} : Y_{\mathbb{P}} \to \mathbb{P}$ is the projection.

Proof. Since $\hat{p} : Y_{\mathbb{P}} \to \mathbb{P}$ is flat, we have a short exact sequence
$$0 \to \hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \to \hat{p}^*\mathcal{O}_{\mathbb{P}_j} \to \hat{p}^*\mathcal{O}_{\partial\mathbb{P}_j} \to 0.$$
By Lemma 4.9 tensoring with $\tilde{\Delta}_*((\xi_w)_{\mathbb{P}})$ over $\mathcal{O}_{Y_{\mathbb{P}}}$ preserves exactness of the above sequence, so we have an exact sequence
$$0 \to \hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \to \hat{p}^*\mathcal{O}_{\mathbb{P}_j} \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}})$$
$$\to \hat{p}^*\mathcal{O}_{\partial\mathbb{P}_j} \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \to 0.$$
By Theorem 6.2, for general $\gamma \in \bar{\Gamma}$, tensoring the above sequence with $\gamma_*\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}$ over $\mathcal{O}_{\bar{Y}_{\mathbb{P}}}$ preserves exactness, so we have, for general $\gamma \in \bar{\Gamma}$, an exact sequence
$$0 \to \gamma_*\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*(\mathcal{O}_{\mathbb{P}_j}(-\partial\mathbb{P}_j)) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right)$$
$$(12) \qquad \to \gamma_*\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*\mathcal{O}_{\mathbb{P}_j} \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right)$$
$$\to \gamma_*\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*\mathcal{O}_{\partial\mathbb{P}_j} \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right) \to 0.$$

Next, consider the exact sequence

$$(13) \qquad 0 \to \mathcal{O}_{N_\gamma}(-M_\gamma) \to \mathcal{O}_{N_\gamma}(-M_\gamma^2) \to \mathcal{O}_{M_\gamma^1}(-M_\gamma^1 \cap M_\gamma^2) \to 0.$$

Comparing (12) with (13) we see that it is sufficient to show that

$$(14) \qquad \gamma_*\mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*\mathcal{O}_{\mathbb{P}_j} \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right) = \mathcal{O}_{N_\gamma}(-M_\gamma^2)$$

and

$$(15) \qquad \gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^* \mathcal{O}_{\partial \mathbb{P}_j} \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \right) = \mathcal{O}_{M_\gamma^1}(-M_\gamma^1 \cap M_\gamma^2).$$

To prove (14) and (15) consider the short exact sequence

$$(16) \qquad 0 \to \tilde{\Delta}_*((\xi_w)_{\mathbb{P}}) \to \tilde{\Delta}_*(\mathcal{O}_{(X_w)_{\mathbb{P}}}) \to \tilde{\Delta}_*(\mathcal{O}_{(\partial X_w)_{\mathbb{P}}}) \to 0.$$

Tensor the sequence (16) with $\hat{p}^*(\mathcal{O}_{\mathbb{P}_j})$ over $\mathcal{O}_{Y_{\mathbb{P}}}$, which preserves exactness by the proof of Lemma 4.9, and then tensor with $\gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}$ over $\mathcal{O}_{\bar{Y}_{\mathbb{P}}}$, which preserves exactness by Theorem 6.2. For closed subschemes X and Y of a scheme Z,

$$\mathcal{O}_X \otimes_{\mathcal{O}_Z} \mathcal{O}_Y = \mathcal{O}_{X \cap Y}.$$

Thus,

$$\gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*(\mathcal{O}_{\mathbb{P}_j}) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*(\mathcal{O}_{(X_w)_{\mathbb{P}}}) \right) = \mathcal{O}_{\gamma(X^u \times X^v)_{\mathbb{P}} \cap \tilde{\Delta}((X_w)_j)} = \mathcal{O}_{N_\gamma}$$

and

$$\gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}} \otimes_{\mathcal{O}_{\bar{Y}_{\mathbb{P}}}} \left(\hat{p}^*(\mathcal{O}_{\mathbb{P}_j}) \otimes_{\mathcal{O}_{Y_{\mathbb{P}}}} \tilde{\Delta}_*(\mathcal{O}_{(\partial X_w)_{\mathbb{P}}}) \right) = \mathcal{O}_{\gamma(X^u \times X^v)_{\mathbb{P}} \cap \tilde{\Delta}((\partial X_w)_j)} = \mathcal{O}_{M_\gamma^2}.$$

Thus, (14) follows.

Similarly, (15) follows by tensoring the sequence (16) with $\hat{p}^*(\mathcal{O}_{\partial \mathbb{P}_j})$ over $\mathcal{O}_{Y_{\mathbb{P}}}$, which preserves exactness by the proof of Lemma 4.9, and then tensoring with $\gamma_* \mathcal{O}_{(X^u \times X^v)_{\mathbb{P}}}$ over $\mathcal{O}_{\bar{Y}_{\mathbb{P}}}$, which preserves exactness by Theorem 6.2. This completes the proof. $\qquad\square$

By the previous lemma, Theorem 5.1 part b) is equivalent to the following theorem:

Theorem 9.2. *For general $\gamma \in \bar{\Gamma}$ and any $u, v, w \in W, j \in [N]^r$ such that $c_{u,v}^w(j) \neq 0$ we have*

$$H^p(N_\gamma, \mathcal{O}_{N_\gamma}(-M_\gamma)) = 0,$$

for all $p \neq |j| + \ell(w) - \ell(u) - \ell(v)$.

Proof. First, the theorem is equivalent to the statement that for general $\gamma \in \bar{\Gamma}$,

$$(17) \qquad H^p(N_\gamma, \omega_{N_\gamma}(M_\gamma)) = 0, \ \forall p > 0.$$

To see this, observe that:

$$\begin{aligned}
H^p(N_\gamma, \omega_{N_\gamma}(M_\gamma)) &= H^p(N_\gamma, \mathscr{H}om_{\mathcal{O}_{N_\gamma}}(\mathcal{O}_{N_\gamma}(-M_\gamma), \omega_{N_\gamma})) \\
&\overset{\varphi_1}{\simeq} \operatorname{Ext}_{N_\gamma}^p(\mathcal{O}_{N_\gamma}(-M_\gamma), \omega_{N_\gamma}) \\
&\overset{\varphi_2}{\simeq} H^{n-p}(N_\gamma, \mathcal{O}_{N_\gamma}(-M_\gamma))^*,
\end{aligned}$$

where $n := \dim N_\gamma = |j| + \ell(w) - \ell(u) - \ell(v)$, the isomorphism φ_1 follows by Corollary 7.11 and the local to global Ext spectral sequence [Go, Théorème 7.3.3, Chap. II], and the isomorphism φ_2 follows by Corollary 7.10 and Serre duality [H, Chap. III, Theorem 7.6].

We now prove (17), which implies the theorem. By [S, Chapter I, §6.3, Theorem 1.25] and [H, Chap. III, Exercise 10.9], there is a non-empty open subset $\mathcal{U} \subset \bar{\Gamma}$ such that $\pi : \pi^{-1}(\mathcal{U}) \to \mathcal{U}$ is flat. (Observe that, by the proof of Corollary 7.10, π is surjective.) We prove that $\omega_{\mathcal{Z}}(\partial \mathcal{Z})$ is flat over $\mathcal{U}$:

To show this, let $A = \mathcal{O}_{\mathcal{U}}$, $B = \mathcal{O}_{\pi^{-1}(\mathcal{U})}$, and $M = \omega_{\mathcal{Z}}(\partial \mathcal{Z})|_{\pi^{-1}(\mathcal{U})}$. By taking stalks, we immediately reduce to showing that for an embedding of local rings

$A \subset B$ such that A is regular and B is flat over A, we have that M is flat over A. Now, to prove this, let $\{x_1, \ldots, x_d\}$ be a minimal set of generators of the maximal ideal of A. Let $K_\bullet = K_\bullet(x_1, \ldots, x_d)$ be the Koszul complex of the x_i's over A. Then, recall that a finitely generated B-module N is flat over A iff $K_\bullet \otimes_A N$ is exact except at the extreme right, i.e., $H^i(K_\bullet \otimes_A N) = 0$ for $i < d$ [E, Corollary 17.5 and Theorem 6.8]. Thus, by hypothesis, $K_\bullet \otimes_A B$ is exact except at the extreme right and hence the x_i's form a B-regular sequence [E, Theorem 17.6]. Now, since $\mathcal{O}_{\mathcal{Z}}$ and $\mathcal{O}_{\partial \mathcal{Z}}$ are CM, we have that $\mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z})$ is a CM $\mathcal{O}_{\mathcal{Z}}$-module. Thus, by [I, Proposition 11.33], we have that M is a CM B-module of dimension equal to $\dim B$. Therefore, by [I, Exercise 11.36], the x_i's form a regular sequence on the B-module M. Hence, $(K_\bullet \otimes_A B) \otimes_B M \simeq K_\bullet \otimes_A M$ is exact except at the extreme right [E, Corollary 17.5]. This proves that M is flat over A, as desired.

Thus, by Corollary 8.8 and the semicontinuity theorem [Ke, Theorem 13.1] to prove (17), it is sufficient to show that for general $\gamma \in \bar{\Gamma}$, that

$$(18) \qquad \omega_{\mathcal{Z}}(\partial \mathcal{Z})|_{\pi^{-1}(\gamma)} \simeq \omega_{N_\gamma}(M_\gamma).$$

To prove this, observe that since $\mathcal{U}$ is smooth and $\mathcal{Z}$ and $\partial \mathcal{Z}$ are CM, and the assertion is local in $\mathcal{U}$, it suffices to observe (cf. [I, Corollary 11.35]) that for a non-zero function θ on $\mathcal{U}$, the sheaf

$$\mathcal{S}/\theta \cdot \mathcal{S} = \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}_\theta}}(\mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z})/\theta \cdot \mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \omega_{\mathcal{Z}_\theta}),$$

where $\mathcal{Z}_\theta$ denotes the zero scheme of θ in $\mathcal{Z}$ and the sheaf

$$\mathcal{S} := \mathcal{H}om_{\mathcal{O}_{\mathcal{Z}}}(\mathcal{O}_{\mathcal{Z}}(-\partial \mathcal{Z}), \omega_{\mathcal{Z}}).$$

Choosing θ to be in a local coordinate system and using induction and the above result, the desired conclusion (18) is obtained. $\qquad \square$

Acknowledgments

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