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Random operators: disorder effects on quantum spectra and dynamics, by Michael Aizenman and Simone Warzel, Graduate Studies in Mathematics, Vol. 168, American Mathematical Society, 2015, xiv+326 pp., ISBN 978-1-4704-1913-4, US $79.00. Individual member price $63.20.

Anderson localization is one of the most important and fascinating phenomena arising from quantum theory. Physically, it is best described by the fact that the introduction of disorder into a quantum system leads to a strong tendency towards the absence of wave transport. This explains theoretically why the presence of random impurities in a conducting crystalline metal will generally turn it into an electrical insulator. Every realistic system exhibits some degree of disorder, so that any quantum theory has to be tested towards the possibility of Anderson localization, leading to the omnipresence of the latter in physics.

The fascination with Anderson localization has reached deep into mathematics. Properly describing the associated phenomena and finding rigorous ways to prove them, starting in the late 1970s, has broadly transformed large parts of mathematical spectral and scattering theory. It has led to the development of heavy machinery, but also required deep and elegant insights from many mathematical fields. The vast literature does not allow a comprehensive review, but the 399-item bibliography in Aizenman and Warzel's new book provides an up-to-date starting point for surveying what has been done.

The prototypical object of study is the Anderson model

$$H = T + \lambda V \tag{1}$$

on $\ell^2(\mathbb{Z}^d)$. That is, T is the adjacency operator (or, up to a shift, discrete Laplacian)

$$(T\psi)(x) := \sum_{y:d(x,y)=1} \psi(y), \tag{2}$$

and $(V\psi)(x) = V(x)\psi(x)$ is a random potential given by i.i.d. random variables $(V(x))_{x\in\mathbb{Z}^d}$ whose joint distribution we will assume to have a bounded density ρ on $\mathbb{R}$. Here $d(x,y) = \sum_j |x_j - y_j|$ is the ℓ^1-distance and $\lambda > 0$ is a disorder parameter. Assuming that ρ has bounded support is the simplest way to assure that the random operator H is almost surely self-adjoint.

Anderson localization of H in an interval I of its spectrum is now generally understood to mean all of the following:

(i) *Spectral localization.* H has almost surely dense pure point spectrum in I with exponentially decaying eigenfunctions.
(ii) *Dynamical localization.* Solutions ψ_t of the time-dependent Schrödinger equation $d\psi_t/dt = -iH\psi_t$ remain well localized in $\mathbb{Z}^d$ if ψ_0 is localized.
(iii) *Poisson statistics of eigenvalues.* The suitably rescaled eigenvalues of the restrictions $H_L = \chi_{\Lambda_L} H \chi_{\Lambda_L}$ of H to finite boxes $\Lambda_L = [-L,L]^d \cap \mathbb{Z}^d$ converge to a Poisson process as $L \to \infty$ (here χ_{Λ_L} denotes the multiplication operator by the characteristic function of Λ_L).

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It has been rigorously proven that H is localized in all of these forms in the following three regimes: (i) in the entire spectrum and for any dimension d if the disorder λ is sufficiently large, (ii) in the entire spectrum if $d = 1$ for any $\lambda > 0$, including small disorder, (iii) near band edges of the spectrum for arbitrary d and any $\lambda > 0$.

Two important physical conjectures for the Anderson model remain open, despite strong efforts and some partial and related results: (i) existence of a mobility edge in dimensions $d \geq 3$ for small disorder λ, i.e., the existence of extended states (characterized either spectrally by the existence of absolutely continuous spectrum or dynamically by the existence of quantum transport) near the centers of spectral bands, separated from the localized band edges; (ii) non-existence of a mobility edge in $d = 2$, i.e., the spectrum is fully localized at arbitrarily small disorder.

The mathematics (as well as the physics) of Anderson localization is very different in the one-dimensional and multidimensional settings. The first proofs of localization in the late 1970s and early 1980s were for one-dimensional models, and they relied strongly on dynamical systems methods; see the monographs [8] and [18] for detailed accounts.

The method of *multiscale analysis* (MSA), as introduced by Fröhlich and Spencer [11] in 1983, was historically the first approach which provided rigorous proofs of localization for the multidimensional Anderson model. In addition to a number of original research papers which developed different versions of the MSA technique, there are several books and book-length papers which give a detailed introduction to this method, including the two books mentioned above. While mastering the machinery at the core of MSA requires substantial effort, two of the more recent presentations in [21] (covering the continuum Anderson model) and [16] (for the lattice case) give very good introductions into this method and are accessible to motivated newcomers.

A second method to prove multidimensional Anderson localization was introduced in 1993 by Aizenman and Molchanov [4] and is now generally referred to by the names of its authors or as the *fractional moments method* (FMM). While MSA has proven to be less technically restrictive and applicable to a broader range of models (including, for example, models of quasi-crystals), the FMM approach has two fundamental advantages: it is much more elementary, at least for the case of lattice models, and leads to the strongest physically expected results for random systems.

It is surprising that more than two decades passed since the introduction of the FMM before Aizenman and Warzel have now provided a comprehensive treatment of the methods, results, and applications following from this approach to localization. As a result, many of these topics have appeared now for the first time in book form (while partial accounts of the FMM have been provided in other books or book chapters; e.g., [9, 19, 22]). Given that the fractional moments method provides a very natural and direct entry into understanding Anderson localization, this book has been long awaited and will undoubtedly become an invaluable resource for both researchers and students.

After covering the basic theory of random operators in the first four chapters (relations between spectra and dynamics, ergodic operators, density of states), important connections between Green's function and eigenfunctions are established in Chapter 5, much of it based on elementary finite-one perturbation arguments. In particular, the Simon–Wolff criterion is proven, allowing us to conclude spectral localization from sufficiently rapid Green function decay. These tools are crucial and heavily used in later parts of the book.

In Chapter 6 the authors move on to the core of their subject by providing a proof of localization of the Anderson model at large disorder. I find it worthwhile to include a detailed discussion of this highly elegant argument here, mainly to support the above claim of simplicity of the FMM. Much of this reasoning seems to have first appeared in [13]. A more leisurely reading may skip this part and proceed with (10) below.

The Green function of the operator H from (1) is $G(x,y;z) = \langle \delta_x, (H-z)^{-1}\delta_y \rangle$, where $z \in \mathbb{C}^+$ and δ_x, δ_y are standard basis vectors in $\ell^2(\mathbb{Z}^d)$. We will initially consider the restriction $H_L = \chi_{\Lambda_L} H \chi_{\Lambda_L}$ of H to the finite box $\Lambda_L = [-L,L]^d \cap \mathbb{Z}^d$ and its Green function G_L.

An iterated application of the resolvent identity

$$(T+\lambda V - z)^{-1} - (\lambda V - z)^{-1} = -(\lambda V - z)^{-1} T (T + \lambda V - z)^{-1}$$

yields the random walk expansion

$$G_L(x,y;z) = \sum_{\gamma : x \mapsto y} (-1)^{|\gamma|} \prod_{k=0}^{|\gamma|} \frac{1}{V(\gamma(k)) - z}, \tag{3}$$

assuming $\operatorname{Im} z > \|T\|$ to assure convergence. Summation is over all finite paths $\gamma = (\gamma(0) = x, \gamma(1), \ldots, \gamma(|\gamma|) = y)$ from x to y, their length denoted by $|\gamma|$. To each path γ corresponds a self-avoiding path $\hat{\gamma}$ found by erasing all loops from γ. Partial resummation in (3) over paths γ with common loop-erased $\hat{\gamma}$ yields the Feenberg expansion [10]

$$G_L(x,y;z) = \sum_{\substack{\text{SAW} \\ \hat{\gamma}: x \mapsto y}} (-1)^{|\hat{\gamma}|} \prod_{k=0}^{|\hat{\gamma}|} \left\langle \delta_{\hat{\gamma}(k)}, \frac{1}{H_{\hat{\gamma},k} - z} \delta_{\hat{\gamma}(k)} \right\rangle, \tag{4}$$

where summation is over self-avoiding paths from x to y, and $H_{\hat{\gamma},k}$ is the restriction of H to $\Lambda_L \setminus \bigcup_{j<k} \{\hat{\gamma}(j)\}$. Note that this sum is finite, so that (4) extends to all $z \in \mathbb{C}^+$ by analytic continuation. A rank-one perturbation argument shows that

$$\left\langle \delta_{\hat{\gamma}(k)}, \frac{1}{H_{\hat{\gamma},k} - z} \delta_{\hat{\gamma}(k)} \right\rangle = \frac{1}{\lambda V(\hat{\gamma}(k)) - \Gamma_{0,k}(z)} \tag{5}$$

with a number $\Gamma_{0,k}(z)$ which is independent of the random variable $V(\hat{\gamma}(k))$. Thus, for $0 < s < 1$, we can bound the conditional expectation

$$\mathbb{E}\left(\left| \left\langle \delta_{\hat{\gamma}(k)}, \frac{1}{H_{\hat{\gamma},k} - z} \delta_{\hat{\gamma}(k)} \right\rangle \right|^s \mid V_{\neq \hat{\gamma}(k)} \right) \le \sup_{\Gamma \in \mathbb{C}} \int \frac{\rho(v)}{|\lambda v - \Gamma|^s} dv = \frac{C_s}{|\lambda|^s} < \infty \tag{6}$$

and, iteratively,

$$\mathbb{E}\left(\prod_{k=0}^{|\hat{\gamma}|} \left| \left\langle \delta_{\hat{\gamma}(k)}, \frac{1}{H_{\hat{\gamma},k} - z} \delta_{\hat{\gamma}(k)} \right\rangle \right|^s \right) \le \left(\frac{C_s}{|\lambda|^s} \right)^{|\hat{\gamma}|+1}. \tag{7}$$

Here close attention needs to be paid to the order in which integrations are carried out and to the random parameters which $H_{\hat{\gamma},k}$ does (or does not) depend on. This is the main trick behind the iterative decoupling of random parameters leading to (4).

The number of self-avoiding paths of length k with given endpoints is bounded by $(2d)^{k-1}$, so that (4) leads to

$$\mathbb{E}\left(|G_L(x,y;z)|^s\right) \leq \sum_{k=d(x,y)}^{\infty} (2d)^{k-1} \left(\frac{C_s}{|\lambda|^s}\right)^{k+1} \leq Ce^{-\mu d(x,y)} \tag{8}$$

if $e^{-\mu} := 2dC_s/|\lambda|^s < 1$, with suitable $C = C(s,\lambda,d)$. Note that here the large disorder assumption on λ has entered. We also used that

$$\left(\sum_j |a_j|\right)^s \leq \sum_j |a_j|^s \tag{9}$$

for arbitrary a_j.

The constants in (8) are independent of L, so that one may use strong resolvent convergence of H_L to H and Fatou's lemma to extend the bound to infinite volume,

$$\mathbb{E}\left(|G(x,y;z)|^s\right) \leq Ce^{-\mu d(x,y)} \tag{10}$$

for all $x, y \in \mathbb{Z}^d$ and $z \in \mathbb{C}^+$ (the case $x = y$ needs another reference to the a priori bound (6)).

Thus we have arrived at exponential decay of fractional moments of the Green function (10). This type of bound is the origin of the name of the FMM and by itself is now among the well-accepted manifestations of Anderson localization. Another manifestation, that H almost surely has pure point spectrum with exponentially decaying eigenfunctions, follows from it by the Simon–Wolff criterion [20].

We point out that the random walk expansion (4) was already used in the argument of Anderson's seminal paper [5]. Thus the fractional moments approach to Anderson localization can be seen as a way to rigorously implement ideas which have been around in the physics of Anderson localization from the very beginning. The FMM approach provides a very transparent and painless way to deal with the arising small denominator problem when the spectral parameter z is close to the real line. In fact, due to working with fractional powers $0 < s < 1$, one could directly integrate through the singularities in (6), meaning that in the resulting finite-volume Green function bound (8) the spectral parameter z can be chosen real.

The above also demonstrates that a key for the simplicity of FMM is to work with bounds on *expectations* throughout. This is much more straightforward than dealing with the small denominators by bounding probabilities of resonant sets where the Green function is close to a singularity, as done in the KAM-type inductive scheme of MSA, where the ultimate aim are *almost sure* bounds.

Another important topic to discuss here is the key role of *eigenfunction correlators* in localization theory. Starting from the mid-1990s, first in [2], they have been understood to be not only the correct mathematical object to consider, but also to express localization in all its forms desired for physical applications. Aizenman and Warzel discuss eigenfunction correlators (EFC) systematically in Chapters 7 to 12, as well as their use to prove Anderson localization in all the physically expected regimes. For a Borel set $I \subset \mathbb{R}$, the EFC is given as (recall that δ_x denotes a standard basis vector)

$$Q(x,y;I) = \sup_{F \in C(\mathbb{R}), \|F\|_\infty \leq 1} |\langle \delta_x, P_I(H)F(H)\delta_y\rangle|. \tag{11}$$

Here the supremum is taken over continuous functions on $\mathbb{R}$, uniformly bounded in absolute value by 1, $F(H)$ is defined via the functional calculus for self-adjoint operators, and $P_I(H)$ denotes the spectral projection onto I for H. In finite volume Λ_L and under the assumption of nondegeneracy of eigenvalues (valid with probability one for the Anderson model) the corresponding quantity is easily seen to be equal to

$$Q_L(x, y; I) = \sum_{E \in \sigma(H_L) \cap I} |\varphi_E(x)||\varphi_E(y)|, \tag{12}$$

where φ_E is the normalized eigenfunction of H_L to E. Thus Q_L measures the correlation of eigenfunctions at different lattice sites x and y. Heuristics suggest that in a regime of well-localized eigenfunctions, the latter should not be simultaneously large at distant sites, so that $Q(x, y, I)$ should decay rapidly in $d(x, y)$.

This is confirmed by a crucial relation of the EFC to fractional moments of Green's function,

$$\mathbb{E}[Q(x, y; I)] \le C_s \liminf_{\eta \to 0} \int_I \mathbb{E}[|G(x, y; E + i\eta)|^s]\, dE, \tag{13}$$

proven in Chapter 7 by using essentially the same finite-rank perturbation ideas which are behind the Simon–Wolff criterion.

An example is the Anderson model at large disorder, where one readily combines (10) with (13) to get *exponential EFC localization*,

$$\mathbb{E}\left(Q(x, y; I)\right) \le C' e^{-\mu|x-y|}. \tag{14}$$

Here I may be any bounded interval, and, in fact, for a bounded random potential V it may be dropped altogether as the spectrum of H can be covered by a deterministic bounded interval I (so that $Q(x, y; I) = Q(x, y; \mathbb{R})$ in (12)).

Taking the supremum in (11) only over the functions $F_t(x) = e^{-itx}$, $t \in \mathbb{R}$, shows that an immediate consequence of (14) is dynamical localization in the form

$$\mathbb{E}\left(\sup_{t \in \mathbb{R}} |\langle \delta_x, P_I(H) e^{-itH} \delta_y \rangle|\right) \le C' e^{-\mu|x-y|}. \tag{15}$$

Before being shown via the FMM, the localization bounds (14) and (15) were known only for the one-dimensional Anderson model (with $I = \mathbb{R}$, independent of disorder strength), where they are a consequence of the method of Kunz and Souillard [17]. A somewhat weaker (subexponential) bound on the decay of eigenfunction correlators has subsequently been derived by an MSA scheme, at the benefit of holding for more singular random parameters. See for example [12] for the case of the *continuum* Anderson model in arbitrary dimension d, which allows for Bernoulli distributed $V(x)$. This is based on a version of MSA in the continuum allowing Bernoulli variables derived by Bourgain and Kenig [7] (which has resisted adaptation to the lattice case due to a lack of required unique continuation properties of the eigenfunctions of (1)). More recently, Imbrie has introduced a new form of the MSA scheme in [15] and argued that this leads to the exponential EFC bound (14).

The Green function localization bound (10) and EFC localization (14) have proven to be powerful and flexible tools in applications. Two such physical applications, the vanishing of the direct conductance of an electron gas (as described by the Kubo formula) and the quantization of the quantum Hall conductance at zero temperature if the Fermi energy E_F lies in the localized regime, are discussed by Aizenman and Warzel in Chapters 13 and 14, based on [3]. Without trying to go into detail, we mention that localization of the Fermi projection $\langle \delta_x, P_{(-\infty,E_F)}(H)\delta_y \rangle$, as covered by EFC localization, is an important ingredient.

A common theme of the final three Chapters 15 to 17 are the surprisingly rich spectral properties of the Anderson model on the Bethe lattice, i.e., the infinite tree graph of constant vertex degree. For this, note that the proof of (10) above extends immediately to arbitrary graphs with bounded vertex degree, in particular the Bethe lattice, proving localization at sufficiently large disorder (the traditional MSA scheme does not work here due to the exponential volume growth of such graphs). On the other hand, the Anderson model on the Bethe lattice is a rare example of a random operator with extensive disorder where the existence of regions of absolutely continuous spectrum can be shown for small disorder. This has been proven by several methods for energies near the spectrum of the unperturbed adjacency operator, but also includes the recent addition of a *resonant delocalized regime* to the phase diagram. This gives regions near the spectral edges which are absolutely continuous despite smallness of the density of states, a phenomenon which was overlooked in the physics literature. Another surprise is that the resonant delocalized regime on tree graphs violates the *spectral statistics conjecture* for the Anderson model. As discussed in Chapter 17, this relates the spectral type of the infinite volume Anderson model to the statistical distribution of eigenvalues in the finite volume model, with pure point spectrum corresponding to Poisson statistics and absolutely continuous spectrum corresponding to the GOE/GUE eigenvalue processes appearing in random matrix theory. The resonant delocalized regime for the Anderson model on the Bethe lattice, however, combines absolutely continuous spectrum with Poisson statistics!

We conclude by remarking that the physics of disordered quantum systems continues to provide vast challenges for mathematics. In the most recent decade, the physics literature has seen an exploding rise of interest in disordered many-body systems, with one of the groundbreaking works given by [6]. The central question is to understand how localization phenomena are changed by the presence of particle interactions. Known results on Anderson localization, such as those described above, have helped us understand the localization properties of some very simple interacting many-body systems; see for example the recent survey [1] for rigorous results on the disordered XY spin chain (as well as for more references). Mathematical approaches to localization in more general many-body models, such as [14], are still rare and incomplete.

Thus the toolbox provided in works such as the book by Aizenman and Warzel will stay relevant as the basis for much further research and will have to find many more applications and extensions.

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