BOOK REVIEWS

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Optimal stochastic control, stochastic target problems, and backward SDE, by N. Touzi, Fields Institute Monographs, Vol. 29, Springer, New York, Fields Institute for Research in the Mathematical Sciences, Toronto, ON, 2013, x+214 pp., ISBN 978-1-4614-4286-8

This is an excellent book on the topic of Stochastic Control Problems (**SCP**). The author transformed his notes for a graduate course at the Field Institute into a volume that will serve also as a good reference in the area. The exposition of the material is done with a modern approach and some emphasis in applications, mainly in mathematical finance. The author has chosen the framework of diffusions, which makes the exposition more friendly and accessible to a larger audience, in particular for those who want to learn this topic.

The book is divided into 13 chapters. In this reviewer's view, the most interesting are chapters 3 through 9. The author also includes two chapters (10–11) on Backward Stochastic Differential Equations (**BSDE**), two chapters (12–13) on numerical aspects of the material presented and an introductory/survey Chapter 2 on Stochastic Differential Equations (**SDE**) and the Black-Sholes model.

The author often introduces new concepts under stronger than necessary regularity conditions, in order to make the main ideas clear. After that more general results and constructions are presented. This is a good idea and tells the reader where he/she should focus to learn this material. Before I continue to describe the topic of this book, I would also like to say that the bibliography is a bit too brief. I have added a few references (see [1]–[10]) to books on the subject that either complement the one under review or have a common point of view with it.

The main objective in SCP is the study of existence, uniqueness, and regularity of solutions for the optimization problem

$$(\mathbf{SCP}) \qquad V(t,x) = \sup_{\nu\in\mathcal{U}_0} J(t,x,\nu) \ \text{ for } (t,x)\in\mathbf{S},$$

where ν is a control variable,

$$J(t,x,\nu) = \mathbb{E}\left(\int_t^T \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s)\, ds + \beta^\nu(t,T) g(X_T^{t,x,\nu})\right),$$

$$\beta^\nu(t,s) = \exp\left(-\int_t^s k(r, X_r^{t,x,\nu}, \nu_r)\, dr\right),$$

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and $X_\bullet^{t,x,\nu}$ solves the SDE in $[t,T]$, driven by the n-dimensional Brownian motion W,

$$(\mathbf{SDE}) \quad dX_s = b(s, X_s, \nu_s)\, ds + \sigma(s, X_s, \nu_s)\, dW_s, \text{ with initial condition } X_t = x.$$

Special attention is paid to describing the local behavior of V using the **Dynamic Programming Equation**, also known as the **Hamilton-Jacobi-Bellman** equation.

The coefficients of this problem are called different names depending on the custom in each area of application: b is called the drift of X, σ the diffusion matrix, k the discounting rate. The functional J is called the **gain function**. $\mathcal{U}_0$ is the set of admissible controls, and T is the horizon time or maturity time. For the sake of simplicity, in this exposition we shall assume that T is a finite fixed time.

We denote by $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ the natural filtration of W, that is, $\mathcal{F}_t$ is the σ-field generated by $(W_u : 0 \leq u \leq t)$. This is the way probabilists model the increasing acquisition of information. A process $(X_t : t \geq 0)$ is said to be **adapted** if for all $t \geq 0$, $X_t \in \mathcal{F}_t$; and **progressively** measurable if for all $t \geq 0$ the function $(s, \omega) \to X_s(\omega)$, $s \leq t$, is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.

Of course, one has to impose some assumptions to make this problem meaningful. In particular, it is standard to assume the following set of conditions on the coefficients involved in (**SCP**).

(A1) The set of controls $\mathcal{U}$ consists of all the progressively measurable processes $\nu = (\nu_t : t \leq T)$ with values in a subset $U \in \mathbb{R}^m$. $\mathcal{U}_0$ is the subset of $\mathcal{U}$ consisting of the square integrable processes ν, that is,

$$\mathbb{E}\left(\int_0^T |\nu_t|^2\, dt\right) < \infty.$$

(A2) b, σ are continuous functions, defined in $\mathbf{S} \times U$, that satisfy the Lipschitz and growth conditions

$$|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \leq K|x-y|;$$

$$|b(t,x,u)| + |\sigma(t,x,u) \leq K(1 + |x| + |u|).$$

(A3) $f, k : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ are continuous functions, $g : \mathbb{R}^n \to \mathbb{R}$ is a measurable function satisfying the growth conditions

k^-, the negative part of k, is bounded and
$|f(t,x,u)| + |g(x)| \leq K(1 + |x|^2 + |u|)$.

It is standard to show that under assumptions $\mathbf{A1}, \mathbf{A2}$, for all $(t,x) \in \mathbf{S}$, there is a unique adapted square integrable process $(X_s^{t,x,\nu} : s \in [t,T])$ solution to (**SDE**). In addition, under (**A3**) the gain function J is well defined.

Finally, here are some useful notions about controls.

(i) An **optimal control** $\widehat{\nu} \in \mathcal{U}_0$ is a solution to (**SCP**), namely $V(t,x) = J(t,x,\widehat{\nu})$.
(ii) A control is said to be a **feedback control** if it is adapted to the filtration generated by X.
(iii) A control of the form $\nu_s = h(s, X_s)$, for some measurable function h, is called a **Markovian control**.
(iv) Finally, a deterministic control is called an **open loop control**.

1. Dynamic programming principle

The dynamic programming principle (**DPP**) is one of the main ideas in the theory of stochastic control. This principle, which is widely used in optimization and computer science, is simply a way to describe the solution to (**SCP**) in a recursive manner. One can use this description to obtain an equation that V should satisfy. The main problem then is to show that V is actually a solution (even in a weak sense).

An important reduction on the problem, which is somehow crucial, is that for computing $V(t,x)$ we can use $\mathcal{U}_t = \{\nu \in \mathcal{U}_0 : \nu \text{ is independent of } \mathcal{F}_t\}$. This is a consequence of the independence of the increments of W and the strong uniqueness of (**SDE**) (see the argument on page 24 of the book under review).

The **DPP** is obtained as follows. Take any stopping time τ that is independent of $\mathcal{F}_t$ and takes values in $[t,T]$. Then, we have

$$\begin{aligned} J(t,x,\nu) &= \mathbb{E}\left(\int_t^\tau \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s)\, ds + \beta^\nu(t,\tau) J(\tau, X_\tau^{t,x,\nu}, \nu)\right) \\ &\le \mathbb{E}\left(\int_t^\tau \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s)\, ds + \beta^\nu(t,\tau) V(\tau, X_\tau^{t,x,\nu})\right), \end{aligned} \tag{1.1}$$

which implies the inequality

$$V(t,x) \le \sup_{\nu \in \mathcal{U}_t} \mathbb{E}\left(\int_t^\tau \beta^\nu(t,s) f(s, X_s^{t,x,\nu}, \nu_s)\, ds + \beta^\nu(t,\tau) V(\tau, X_\tau^{t,x,\nu})\right).$$

The converse inequality, in general, is not true. If V is continuous, we get the desired inequality by using the first equality in (1.1). Indeed, take any control $\mu \in \mathcal{U}_t$ and any $\epsilon > 0$. Consider $\nu^\epsilon \in \mathcal{U}_\tau$ an ϵ optimal control for $V(\tau, X_\tau^{t,x,\mu})$, that is $V(\tau, X_\tau^{t,x,\mu}) - \epsilon \le J(\tau, X_\tau^{t,x,\mu}, \nu^\epsilon)$. If we denote by ν the control given by μ in the interval $[t,\tau)$ and ν^ϵ on the interval $[\tau, T]$ then

$$\begin{aligned} V(t,x) &\ge J(t,x,\nu^\epsilon) \\ &\ge \mathbb{E}\left(\int_t^\tau \beta(t,s) f(s, X_s, \nu_s^\epsilon)\, ds + \beta(t,\tau) V(\tau, X_\tau)\right) - \epsilon \mathbb{E}(\beta(t,\tau)), \end{aligned}$$

where $X = X^{t,x,\nu}, \beta = \beta^\nu$. This leads to the **DPP**. When V is not that regular, the above argument fails, mainly because measurability and measure zero matters.

With more regularity, we obtain that V solves the dynamic programming equation (**DPE**), which corresponds to Euler's equation in the deterministic case. For simplicity, we assume that V is $C^{1,2}$ with bounded gradient and Hessian. We further assume that k is also bounded.

Consider a fixed control $u \in U$, then we have from the **DPP**

$$\zeta(\rho,h) := V(t,x) - \mathbb{E}\left(\int_t^\tau \beta^u(t,\tau) f(s, X_s, u)\, ds + \beta^u(t,\tau) V(\tau, X_\tau)\right) \ge 0,$$

where $X = X^{t,x,\tau}$ and $\tau = \inf\{s > t : \ |X_s - x| > \rho\} \wedge h$, for $\rho > 0$ and some small $h > 0$. Using Itô's formula, we obtain

$$\zeta(\delta,h) = -\mathbb{E}\left(\int_t^\tau \beta^u(t,\tau)\left[\partial_t V - kV + b \cdot D_x V + \frac{1}{2}\operatorname{Tr}(\sigma\sigma' D_x^2 V) + f\right] ds\right) \ge 0$$

because the local martingale part drops out due to the hypothesis made on the coefficients and V. Passing to the limit in $h \downarrow 0$, we get the inequality

$$-\partial_t V - \mathcal{L}^u V - f \ge 0,$$

where $\mathcal{L}^u(\psi) = -k\psi + b \cdot D_x\psi + \frac{1}{2}\operatorname{Tr}(\sigma\sigma' D_x^2\psi)$ is the generator of the discounted process $\beta^u(t,\bullet)X_\bullet^{t,x,u}$. In particular, we obtain that V is a supersolution (in the classical sense) of the equation

$$-\partial_t V - \sup_{u\in U}(\mathcal{L}^u V + f) \geq 0.$$

The following Hamiltonian, defined on $\mathbf{S}\times\mathbb{R}\times\mathbb{R}^n\times\mathcal{S}_n$, where $\mathcal{S}_n$ is the set of symmetric matrices, appears naturally:

$$\begin{aligned} &H(t,x,r,p,M)\\ &\quad= \sup_{u\in U}\left(-k(t,x,u)r + b(t,x,u)\cdot p + \frac{1}{2}\operatorname{Tr}(\sigma\sigma'(t,x,u)M) + f(t,x,u)\right). \end{aligned}$$

With some extra regularity on H, it can be proved that V is a classical solution to the problem

$$\textbf{(DPE)} \qquad -\partial_t V(t,x) - H\left(t,x,V(t,x),D_xV(t,x),D_x^2V(t,x)\right) = 0.$$

The main problem with this approach is that V in general is not regular. There are simple examples, both in the deterministic and stochastic case, where V is not differentiable. A possible way to confront this situation is to generalize the notion of a solution of the (**SCP**). A successful idea is to introduce the notion of a **viscosity solution**. In our setting, this works well, because the Hamiltonian H is monotonically increasing in the Hessian variable, that is,

$$H(t,x,r,p,M) \leq H(t,x,r,p,N)$$

whenever $M \leq N$ in the sense of semidefinite matrices. The key point is that V is a classical supersolution to $-\partial_t V - H(t,x,V,D_xV,D_x^2V) \geq 0$ if and only if
(**VSupS**)

$$-\partial_t\psi(t_0,x_0) - H(t_0,x_0,V(t_0,x_0),D_x\psi(t_0,x_0),D_x^2\psi(t_0,x_0)) \geq 0,$$

$$\forall(t_0,x_0)\in\mathbf{S},\ \forall\psi\in C^2(\mathbf{S}),\ \text{for which } (t_0,x_0) \text{ is a minimum of } V-\psi.$$

This statement can be used as a definition of viscosity supersolution.

A technical detail to be taken into account is that V has to be replaced by its lower and upper semicontinuous envelopes (due to regularity and measurability matters)

$$\begin{aligned} V^*(t,x) &= \limsup_{(t',x')\to(t,x)} V(t',x') \quad \text{upper-semicontinuous envelope;}\\ V_*(t,x) &= \liminf_{(t',x')\to(t,x)} V(t',x') \quad \text{lower-semicontinuous envelope.} \end{aligned}$$

Then V is said to be a viscosity supersolution of (**DPE**) if V_* satisfies (**VSupS**). A similar definition is made for a viscosity subsolution. Of course a viscosity solution is both a viscosity super- and subsolution.

The key step in proving that V is a viscosity supersolution to (**DPE**) is first to obtain a weak version of the (**DPP**). It has the form
(**WDPP**)

$$V(t,x) \geq \sup_{\nu\in\mathcal{U}_t}\mathbb{E}\left(\int_t^{\tau^\nu}\beta^\nu(t,s)f(s,X_s^{t,x,\nu},\nu_s)\,ds + \beta^\nu(t,\tau^\nu)V_*(\tau,X_{\tau^\nu}^{t,x,\nu})\right),$$

where $\{\tau^\nu,\nu\in\mathcal{U}_t\}$ is a family of stopping times independent of $\mathcal{F}_t$ and taking values in $[t,T]$.

Consider now a fixed C^2 function ψ, such that $V_*(t,x)=\psi(t,x)$ and $\psi \le V_*$, for fixed (t,x). Also consider a sequence $(t_n,x_n)\to(t,x)$, such that $V(t_n,x_n)$ converges to $V_*(t,x)$. For $u\in\mathcal{U}$, we take $\tau_n=\tau_n^u=\inf\{s\ge t: |X_s^{t_n,x_n,u}-x_n|>\rho\}\wedge h_n$. We also denote $X^n := X^{t_n,x_n,u}$ and $a_n := V(t_n,x_n)-\psi(t_n,x_n)$, which converges to 0. Then, substituting these quantities in (**WDPP**), we get

$$\begin{aligned}0 &\le \mathbb{E}\left(V(t_n,x_n)-\beta^u(t_n,x_n)V_*(\tau_n,X^n_{\tau^n})-\int_{t_n}^{\tau_n}\beta^u(t_n,s)f(s,X_s^n,u)\,ds\right)\\ &\le a_n+\mathbb{E}\left(\psi(t_n,x_n)-\beta^u(t_n,x_n)\psi(\tau_n,X^n_{\tau^n})-\int_{t_n}^{\tau_n}\beta^u(t_n,s)f(s,X_s^n,u)\,ds\right).\end{aligned}$$

The net effect is that we have changed V,V_* by the smooth function ψ. Once again, one can use Itô's formula to deduce that

$$-\partial_t\psi(t,x)-H(t,x,V(t,x),D_x\psi(t,x),D_x^2\psi(t,x))\ge 0,$$

and V is a viscosity supersolution of equation (**DPE**).

Similarly, it is proven that V is a subsolution, and the conclusion is that V is a viscosity solution to the dynamical problem equation.

2. Backward stochastic differential equations

Backward stochastic differential equations (**BSDE**) play a fundamental role in several areas of stochastic analysis because these equations give stochastic representations in a variety of problems. Let us start with the following simple idea. Assume that $u(t,x)$ is a solution to the semilinear partial differential equation (on $[0,T]\times\mathbb{R}^n$)

$$\partial_t u(t,x)+\frac{1}{2}\Delta_x u(t,x)=-F(t,x,u(t,x),\nabla_x u(t,x)).$$

Also assume that we can use Itô's formula on $[t,T]$ to get

$$u(T,W_T)=u(t,W_t)+\int_t^T\nabla_x u(s,W_s)\,dW_s+\int_t^T\partial_t u(s,W_s)+\frac{1}{2}\Delta_x u(s,W_s)\,ds,$$

which gives a representation of $u(t,x)$ in terms of the final value $g(x)=u(T,x)$:

$$u(t,W_t)=g(W_T)+\int_t^T F(s,W_s,u(s,W_s),\nabla_x u(s,W_s))\,ds-\int_t^T\nabla_x u(s,W_s)\,dW_s.$$

If we define the adapted processes $Y_s=u(s,W_s), Z_s=\nabla_x u(s,W_s)$, then we get that (Y,Z) satisfies the (**BSDE**)

$$Y_t=g(W_T)+\int_t^T F(s,W_s,Y_s,Z_s)\,ds-\int_t^T Z_s\,dW_s.$$

Hence, $u(t,x)=\mathbb{E}\left(Y_t \,\middle|\, W_t=x\right)$ is represented as a functional of Brownian motion. The case we have presented is the so-called Markovian. A general framework is given by the following data: a final random variable $\xi\in\mathcal{F}_T$, and a generator $F(t,\omega,y,z)$ that is predictable. Then, we search for an adapted solution (Y,Z) that solves the equation on $[0,T]$

$$Y_t=\xi+\int_t^T F(s,W,Y_s,Z_s)\,ds-\int_t^T Z_s\,dW_s.$$

The standard assumptions are $\xi\in L^2(\mathcal{F}_T)$ and, uniformly in (t,ω), F has a linear growth and satisfies a Lipschitz condition in (y,z).

When $F = 0$, the unique solution is given by the predictable representation property of Brownian motion, which says that every $L^2(\mathcal{F}_T)$-random variable is a stochastic integral with respect to Brownian motion,

$$\xi = \mathbb{E}(\xi) + \int_0^T \Gamma_s \, dW_s,$$

and so $Y_t = \mathbb{E}(\xi) + \int_0^t Z_s \, dW_s$, $Z_t = -\Gamma_t$ is the unique solution. Notice here that while Y is a continuous process, Z is just measurable (predictable, to be more precise). This idea, plus a fixed point argument shows existence and uniqueness under the standard assumptions.

We will show now the connection between (**BSDE**) and (**SCP**). This resembles the backward ODE satisfied by the Lagrange multiplier (associated to the dynamics) in the deterministic optimal control problem. We will follow here Theorem 10.1 in the book under review, which maybe considered a "verification" result.

The context will be the following (**SCP**): $V(t,x) = \sup_{\nu \in \mathcal{U}_0} \mathbb{E}\left(g(X_T^{t,x,\nu})\right)$. We introduce for every $u \in \mathcal{U}$ the Lagrangian

$$L^u(t,x,y,z) = b(t,x,u)y + \operatorname{Tr}(\sigma(t,x,u)'z)$$

and its maximal operator

$$\ell(t,x,y,z) = \sup_{u \in \mathcal{U}} L^u(t,x,y,z).$$

The main assumption on the coefficients of this problem is that ℓ and g are concave functions of x. Of course, there are other assumptions about growth conditions and smoothness that we do not make explicit. We assume that there exist processes $\hat{X}, \hat{Y}, \hat{Z}, \hat{\nu}$ that satisfy

(i) Forward-Backward equations

$$\begin{aligned} &\hat{X} = X^{t,x,\hat{\nu}} \\ &d\hat{Y}_r = -\nabla_x L^{\hat{\nu}_r}(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r)\, dr + \hat{Z}_r \, dW_r, \ \hat{Y}_T = \nabla_x g(\hat{X}_T). \end{aligned}$$

(ii) $\hat{\nu}$ satisfies the maximum principle

$$L^{\hat{\nu}_r}(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r) = \ell(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r).$$

(iii)

$$\nabla_x L^{\hat{\nu}_r}(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r) = \nabla_x \ell(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r).$$

Then $\hat{\nu}$ is an optimal control, that is $V(t,x) = J(t,x,\hat{\nu})$. The proof takes just a few lines: Consider an arbitrary control $\nu \in \mathcal{U}_0$, and denote by $X := X^{t,x,\nu}$. The concavity assumption on g yields

$$\begin{aligned} J(\hat{\nu}) - J(\nu) &= \mathbb{E}\left(g(\hat{X}_T) - g(X_T)\right) \\ &\geq \mathbb{E}\left((\hat{X}_T - X_T)\nabla_x g(\hat{X}_T)\right) = \mathbb{E}\left((\hat{X}_T - X_T)\hat{Y}_T\right). \end{aligned}$$

Using the Forward-Backward equations satisfied by $\hat{X}, \hat{Y}$ and the forward satisfied by X, we conclude that

$$
\begin{aligned}
&J(\hat{\nu}) - J(\nu)\\
&\quad \geq \mathbb{E}\left(\int_t^T \left[(\hat{b}(s) - b(s))\hat{Y}_s - (\hat{X}_s - X_s)\nabla_x \hat{L}(s) + \operatorname{Tr}(\hat{\sigma}(s) - \sigma(s))'\hat{Z}_s\right] ds\right)\\
&\quad = \mathbb{E}\left(\int_t^T \left[\hat{L}(s) - L(s) - (\hat{X}_s - X_s)\nabla_x \hat{L}(s)\right] ds\right)\\
&\quad = \mathbb{E}\left(\int_t^T \left[\ell(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s) - L(s) - (\hat{X}_s - X_s)\nabla_x \ell(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s)\right] ds\right)\\
&\quad \geq \mathbb{E}\left(\int_t^T \left[\ell(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s) - \ell(s, X_s, \hat{Y}_s, \hat{Z}_s) - (\hat{X}_s - X_s)\nabla_x \ell(s, \hat{X}_s, \hat{Y}_s, \hat{Z}_s)\right] ds\right).
\end{aligned}
$$

The last quantity is nonnegative due to the concavity of ℓ, and the optimality of $\hat{\nu}$ is shown (we have simplified the notation as $\hat{b}(r) = b(r, \hat{X}_r, \hat{\nu}_r), b(r) = b(r, X_r, \nu_r)$, similarly $\hat{\sigma}(r), \sigma(r)$ and $\hat{L}(r) = L^{\hat{\nu}_r}(r, \hat{X}_r, \hat{Y}_r, \hat{Z}_r), L(r) = L^{\nu_r}(r, X_r, \hat{Y}_r, \hat{Z}_r)$).

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