

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
MICHAEL ASCHBACHER AND BOB OLIVER

MR0004042 (2,309c) 20.0X

Brauer, R.; Nesbitt, C.

On the modular characters of groups.

Annals of Mathematics. Second Series **42** (1941), 556–590.

The authors first summarize the theory of modular characters of a finite group \mathfrak{G} as previously developed by them [Univ. of Toronto Studies, Math. Series, no. 4, 1937]. If an element of \mathfrak{G} is defined to be p -regular when its order is prime to p , one of the important results in this theory is that the number of distinct absolutely irreducible modular representations of \mathfrak{G} is equal to the number of classes of conjugate p -regular elements of \mathfrak{G} . The principal problem considered in this paper is the reduction of an ordinary representation of \mathfrak{G} when the coefficients are taken in a field of characteristic p . It is proved that, if $g = p^\alpha g'$, where $(g', p) = 1$ and g is the order of \mathfrak{G} , then an ordinary irreducible representation Z_i of degree $z_i \equiv 0 \pmod{p^\alpha}$ remains irreducible as a modular representation and the character $\zeta^{(i)}$ of Z_i vanishes for all elements of an order divisible by p . The theory of “blocks,” which is essential to the argument, cannot be explained here. As in the ordinary theory, the Kronecker product of two modular representations of \mathfrak{G} yields a new modular representation whose reduction is discussed. It is shown that the number of self-contragredient modular characters is equal to the number of self-reciprocal p -regular classes of \mathfrak{G} . Again, there is a modular theory corresponding to the ordinary theory relating the representations of \mathfrak{G} to those of a subgroup \mathfrak{H} . As would be expected, the analogy is very close. The paper concludes with a number of special cases and applications of the theory to $GLH(2, p^\alpha)$, $SLH(2, p^\alpha)$ and $LF(2, p^\alpha)$.

G. de B. Robinson

From MathSciNet, July 2016

MR0450410 (56 #8704) 20D20

Puig, Luis

Structure locale dans les groupes finis.

Société Mathématique de France. Bulletin. Supplément. Mémoire. **47** (1976),
132 pp.

The study of a finite group G in terms of its p -subgroups and their normalizers has come to be known as the “local study” of G . In this memoir the author sets out a systematic framework for this study and, working within this framework, proves some new theorems concerning the relation between the local and global structures of G .

The author’s basic definition is that of a p -locality in G . This is a pair (\mathfrak{A}, W) consisting of a nonempty set \mathfrak{A} of p -subgroups of G and a map W associating with each A in \mathfrak{A} a subgroup $W(A)$ of the normalizer $N(A)$, in such a way that if $A \in \mathfrak{A}$, $x \in G$, and B is a p -subgroup of G containing A^x , then $B \in \mathfrak{A}$ and $W(B) \subseteq W(A)^x$. Examples with $\mathfrak{A} =$ all p -subgroups of G include the 1-locality ($W(A) = 1$), and

the C -locality ($W(A) = C(A)$). Associated to (\mathfrak{A}, W) is a category \mathfrak{A}_W , in which the objects are the elements of \mathfrak{A} , a morphism of A into B is a coset $xW(A)$ for which $A \subseteq B^x$, and composition of morphisms is given by multiplication in G . Two p -localities are said to be equivalent if their associated categories are equivalent. If H is a subgroup of G containing some element of \mathfrak{A} , there is a natural definition of the p -locality in H obtained from (\mathfrak{A}, W) by restriction, and H is called an (\mathfrak{A}, W) -control subgroup if this is equivalent to (\mathfrak{A}, W) .

These definitions appear to cover much of what is usually understood to be the local study of G . Examples: (a) A theorem of Frobenius may be stated: G has a normal p -complement if its C -locality is equivalent to that of a p -group. (b) A normal subgroup T of a Sylow p -subgroup P of G controls strong fusion in P with respect to G if and only if $N(T)$ is a control subgroup relative to the C -locality. Strongly embedded subgroups, signalizer functors and the notion of p -constraint also arise in the theory.

The author develops a complex and detailed general theory of p -localities. His main tools are a calculus of “double arrows” (an idea from the fusion theorem of J. L. Alperin [J. Algebra **6** (1967), 222–241; MR0215913]), and the notion of an essential p -subgroup of G (a p -subgroup of G for which $N(A)/A$ is p -isolated [cf. D. M. Goldschmidt, *ibid.* **16** (1970), 138–142; MR0260869]). In addition he studies some particular p -localities, and shows how it is sometimes possible to relate a p -locality to a q -locality ($p \neq q$), and also to relate two different p -localities.

Applications of the theory include a criterion for the existence of a normal π -complement in a group, where π is a set of primes, generalizing the theorem of Frobenius. Also the author defines a characteristic subgroup $L(P)$ of any p -group P , resembling the Thompson subgroup $J(P)$ and the Glauberman subgroup $K_\infty(P)$, and shows that, if $p \neq 2$ and P is a Sylow p -subgroup of G , then G has a normal p -complement if $N(Z(L(P)))$ has.

{Reviewer’s remark: In another context, the idea of making p -subgroups into a category has also been used in work of D. Quillen on cohomology rings [Ann. of Math. (2) **94** (1971), 549–572; MR0298694].}

W. J. Wong

From MathSciNet, July 2016

MR0541333 (80f:20010) 20C99

Alperin, J.; Broué, Michel

Local methods in block theory.

Annals of Mathematics. Second Series **110** (1979), no. 1, 143–157.

The authors develop concepts for a local block theory which is, roughly speaking, a synthesis of local group theory and of generalizations of R. Brauer’s theory of pairs (P, b) [Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), pp. 103–130, Lecture Notes in Math., Vol. 372, Springer, Berlin, 1974; MR0352238]. Instead of special pairs considered by Brauer, they use all subpairs of the form (P, b_P) , where P denotes a p -subgroup of a fixed finite group G and b_P a p -block of $C_G(P)$. Carrying over from group theory the concepts of subgroup, normal subgroup and Sylow subgroup to the language of subpairs, the authors prove analogues of the Sylow theorems and a generalization of Brauer’s third main theorem. In a further section the concept of local fusion in group theory is generalized to the context of subpairs. This entails

a counterpart of the famous fusion theorem of the first author. As a corollary they show that the fusion of subsections in the sense of Brauer is completely determined by local subgroups. For major subsections the authors state a complete set of representatives for G -conjugacy classes. The concept of fusion control subgroups yields a generalization of a classical result of Burnside.

Details cannot be given here because of the many definitions needed to formulate the theorems. The beginning section in which the authors give the background material makes the paper self-contained.

Wolfgang Willems

From MathSciNet, July 2016

MR1709949 (2001i:55017) 55R35; 20D08

Benson, David J.

Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants.

Geometry and cohomology in group theory (Durham, 1994), 10–23, *London Math. Soc. Lecture Note Ser.*, 252, Cambridge University Press, Cambridge, 1998.

The theory of p -compact groups has been developed as a homotopy-theoretical version, or extension, of the theory of compact Lie groups. A p -compact group (in its present definition) is a connected p -complete space whose loop space has finite mod p homology. The motivating example is the p -completion of the classifying space of a compact Lie group whose component group is a p -group. Another example (essentially the only one at $p = 2$) was constructed by W. G. Dwyer and C. W. Wilkerson [*J. Amer. Math. Soc.* **6** (1993), no. 1, 37–64; MR1161306]. This 2-compact group with rational rank three and mod 2 cohomology is given by the rank four Dickson invariants, the fixed subalgebra of the standard action of $\mathrm{GL}(4, \mathbf{F}_2)$ on the polynomial algebra. In many respects this space, which they denote by $B\mathrm{DI}(4)$, is an analogue in rank three of the 2-completion of the classifying space of the exceptional Lie group G_2 .

This paper studies “finite subgroups” of the 2-compact group $B\mathrm{DI}(4)$. For the purposes of this paper such a thing is declared to be a pair (G, i) where G is a finite group and $f: BG \rightarrow B\mathrm{DI}(4)$ is a map rendering $H^*(BG; \mathbf{F}_2)$ a finitely generated module over $H^*(B\mathrm{DI}(4); \mathbf{F}_2)$. In earlier work, it was shown by R. J. Milgram and by the author and Wilkerson that the Mathieu group M_{12} is a subgroup of G_2 regarded as a 2-compact group, despite the fact that M_{12} is not a subgroup of the Lie group G_2 . In this paper a proof is sketched of the analogous fact that the Conway group Co_3 is a subgroup of $B\mathrm{DI}(4)$. In effect, $B\mathrm{Co}_3$ is, up to mod 2 homology equivalence, expressed as a homotopy colimit of a diagram of simpler classifying spaces, and this diagram is related to the diagram used by Dwyer and Wilkerson in their construction. Along the way, certain aspects of the construction of $B\mathrm{DI}(4)$ are clarified.

The “Lang square” [E. M. Friedlander, *Topology* **15** (1976), no. 1, 87–109; MR0394660] exhibits the p -completion of the classifying space of the Chevalley group $G(q)$ associated to a semisimple algebraic group G as a pullback. The vertices of this square make good sense for the 2-compact group $B\mathrm{DI}(4)$, and so its “Chevalley forms” are defined for q an odd prime power. The author points out that the diagram of centralizers of elementary abelian subgroups of these spaces (suitably interpreted) was considered earlier, by R. Solomon [*J. Algebra* **28** (1974),

182–198; MR0344338] in his proof of uniqueness of Co_3 . Solomon concluded that these exotic fusion patterns were not associated to any actual group, by considering the interaction with primes other than 2; but as it turns out they are fusion patterns in some more local sense. This may be taken as evidence for the existence and richness of a “local group theory”, in which the phenomenon of fusion is freed from the global constraints imposed by an ambient group. From a different perspective, this would amount to a theory of “discrete p -compact groups” improving on the current provisional narrow class of p -groups.

Haynes R. Miller

From MathSciNet, July 2016

MR2302494 (2008e:55021) 55R35; 20D20, 20J99, 55S15

Broto, Carles; Levi, Ran; Oliver, Bob

Discrete models for the p -local homotopy theory of compact Lie groups and p -compact groups.

Geometry & Topology **11** (2007), 315–427.

This carefully written paper aims to provide a common context for the theory of p -compact groups established by W. G. Dwyer and C. W. Wilkerson, Jr. [Ann. of Math. (2) **139** (1994), no. 2, 395–442; MR1274096] and the more recent theory of p -local finite groups due to the authors [J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic); MR1992826] (henceforth “BLO2”). The former provides a definition, natural from a homotopy-theoretic perspective, of a class of spaces that includes the p -completions of classifying spaces of compact Lie groups G for which $\pi_0(G)$ is a p -group. BLO2, on the other hand, provides an analogous context for the p -completions of classifying spaces of finite groups, and provides a codification of the local study of group theory.

The present paper incorporates ideas from both these sources. Dwyer and Wilkerson considered the extension of a maximal torus of a Lie group (or a p -compact group) G by a Sylow p -subgroup of its Weyl group, and regarded it as a “Sylow p -subgroup” of G . They also considered a “discrete approximation” of this group, a sub-extension (unique up to conjugation) with kernel given by the p -power torsion of the maximal torus. They defined a discrete p -toral group as an extension of a group of the form $(\mathbb{Z}_{p^\infty})^r$ by a finite p -group. The present paper begins with a useful compendium of properties of such groups, which can also be characterized as Artinian (satisfying the descending chain condition on subgroups) locally finite p -groups (every finitely generated subgroup is a finite p -group). The minimal subgroup of finite index coincides with the subgroup of infinitely p -divisible elements in a discrete p -toral group. It is called the “identity component” of S and is written S_0 . It is isomorphic to $(\mathbb{Z}_{p^\infty})^r$ for a unique integer r , called the rank of S . The order of S is the pair $(\text{rk}(S), |S/S_0|)$, and these are ordered lexicographically. Any subgroup of a discrete p -toral group is again discrete p -toral.

The definition of a p -local finite group began with a “saturated fusion system” on a p -group, abstracting the structure a group imposes on the set of subgroups of any of its Sylow p -subgroups. The first step in the present work is to define the notion of a fusion system on a discrete p -toral group, along with the notions needed to say when it is “saturated”. A fusion system on any group S is a subcategory of the category of subgroups of S and injective homomorphisms between them, that contains conjugations by elements of S and has the property that every morphism

factors as an isomorphism followed by an inclusion. Saturation is a technical condition reflecting the Sylow subgroup theorem, and to the definition found in BLO2 the authors need only add a continuity property.

A saturated fusion system \mathcal{F} on an infinite discrete p -toral group S generally possesses infinitely many isomorphism classes of objects, but the authors define a subcategory \mathcal{F}^\bullet of \mathcal{F} such that the inclusion i has a right adjoint r , ri is naturally equivalent to the identity on \mathcal{F}^\bullet , and \mathcal{F}^\bullet has only finitely many isomorphism classes of objects. Much of the technical work later in the paper takes place in the subcategory \mathcal{F}^\bullet .

There are several important classes of subgroups of S determined by \mathcal{F} . A subgroup $P \leq S$ is \mathcal{F} -centric if every subgroup of S isomorphic in \mathcal{F} to P contains its centralizer in S ; it is fully normalized in \mathcal{F} if the order of its normalizer in S is maximal among orders of S -normalizers of subgroups isomorphic in \mathcal{F} to P ; and it is \mathcal{F} -radical if the finite group $\text{Out}_{\mathcal{F}}(P) = \text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$ contains no nontrivial normal p -subgroups.

Using the subcategory \mathcal{F}^\bullet and the surrounding concepts, the authors prove a version of the Alperin fusion theorem, extending work of J. L. Alperin, L. Puig, D. J. Benson, and earlier work of their own. This theorem guarantees that any isomorphism in \mathcal{F} can be written as a composite of restrictions of conjugations by elements of subgroups $Q \leq S$ which are \mathcal{F} -centric, \mathcal{F} -radical, and fully normalized in \mathcal{F} . This result is used repeatedly in the sequel.

One wishes to construct a “classifying space” associated to a fusion system \mathcal{F} on S . It is natural to attempt to do so using the orbit category $\mathcal{O}(\mathcal{F})$, which is the quotient category obtained by dividing by the right action of $\text{Inn}(Q)$ on $\mathcal{F}(P, Q)$. The classifying space construction gives a functor $B: \mathcal{O}(\mathcal{F}) \rightarrow \text{Ho}(\mathbf{Top})$ into the homotopy category of spaces, and one would like to build a classifying space for \mathcal{F} out of this functor as a homotopy colimit. In order to do so, however, one must first rigidify it, that is, lift it to a functor $\tilde{B}: \mathcal{O}(\mathcal{F}) \rightarrow \mathbf{Top}$. Continuing to follow the trail blazed by BLO2, the authors make a preliminary step by replacing \mathcal{F} by its full subcategory \mathcal{F}^c of \mathcal{F} -centric objects. They then define a purely algebraic structure, a centric linking system associated to \mathcal{F} , such that isomorphism classes of centric linking systems associated to \mathcal{F} are in bijective correspondence with equivalence classes of rigidifications of B (Proposition 4.6). They support this work with an Appendix on lifting diagrams from the homotopy category, which provides a useful interpretation and extension of work of Dwyer and D. M. Kan [Proc. Amer. Math. Soc. **114** (1992), no. 2, 575–584; MR1070515].

A centric linking system associated to a saturated fusion system \mathcal{F} consists of a category \mathcal{L} together with a full functor to \mathcal{F}^c given on objects by the identity map, together with some auxiliary data, and the authors show that the classifying space of \mathcal{L} is weakly equivalent to the homotopy colimit of the corresponding rigidification of B . The classifying space of the linking system is defined to be the p -completion of $|\mathcal{L}|$. The completion process is tractable since $|\mathcal{L}|$ is p -good.

A p -local compact group is defined as a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a discrete p -toral group, \mathcal{F} is a saturated fusion system on S , and \mathcal{L} is a centric linking system associated to \mathcal{F} .

The apparatus of centric linking systems is to some degree justified by Theorem 7.4, which states that the triple $(S, \mathcal{F}, \mathcal{L})$ is determined by the classifying space $|\mathcal{L}|_p^\wedge$. In preparation for this, Section 6 is devoted to studying the space of maps

from the classifying space of a p -group Q to $|\mathcal{L}|_p^\wedge$, re-expressing it as the completed classifying space of a category \mathcal{L}^Q constructed out of Q and $(S, \mathcal{F}, \mathcal{L})$. Considerable information about homotopy automorphisms of $|\mathcal{L}|_p^\wedge$ is obtained in Section 7 as well. Both these sections use extensions to this context of methods for computing the cohomology of functors (that is, their right derived functors) defined on orbit categories, carried out in Section 5.

The remainder of the paper is devoted to examples. Section 8 investigates when one can associate a p -local compact group $(S, \mathcal{F}, \mathcal{L})$ to an infinite discrete group G in such a way that $BG_p^\wedge \simeq |\mathcal{L}|_p^\wedge$. Here is a general result: Theorem 8.7. Suppose that G is a locally finite group which is such that every p -subgroup (all elements are of p -power order) is a discrete p -toral group and such that centralizers of increasing chains of elementary abelian p -subgroups of G stabilize. Then G has a unique conjugacy class of maximal discrete p -toral subgroups, and for any such subgroup S there is a canonically associated p -local compact group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ whose classifying space $|\mathcal{L}_S^c(G)|_p^\wedge$ is weakly equivalent to BG_p^\wedge . This is satisfied, for example, for any torsion subgroup of $\mathrm{GL}_n(k)$ if k is a field of characteristic prime to p .

Section 9 shows that any compact Lie group has a unique conjugacy class of maximal discrete p -toral subgroups, and that for any such subgroup S there is a canonically associated p -local compact group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ whose classifying space is weakly equivalent to BG_p^\wedge . In this example the rigidification of the diagram of classifying spaces is easy to construct using the G -space EG , and the centric linking system itself is left undescribed. Section 10 demonstrates the analogous result for p -compact groups, now using the Dwyer-Kan obstruction theory directly to construct a rigidification, following work of N. Castellana, Levi and D. Notbohm [Adv. in Math. **216** (2007), no. 2, 491–534].

As of the publication of the paper under review, much remains to be done on this project. We quote from the introduction: “One future goal is to show that the mod p cohomology of the classifying space $|\mathcal{L}|_p^\wedge$ of a p -local compact group $(S, \mathcal{F}, \mathcal{L})$ can always be described in terms of the fusion system \mathcal{F} , as a ring of ‘stable elements’ in the cohomology of S . Other goals are to define connected p -local compact groups, and understand their properties and their relation to connected p -compact groups and to characterize algebraically (connected) p -compact groups among all (connected) p -local compact groups. Finally, a more general question which is still open is whether the p -completion of the classifying space of every finite loop space is the classifying space of a p -local compact group.”

Haynes R. Miller

From MathSciNet, July 2016

MR2630058 (2011h:20038) 20D20; 20D08, 20E08, 20J15

Aschbacher, Michael; Chermak, Andrew

A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver.

Annals of Mathematics. Second Series **171** (2010), no. 2, 881–978.

A fusion system \mathcal{F} on a p -group S is called exotic if \mathcal{F} does not equal the fusion system associated to any finite group G with Sylow p -subgroup S . For odd primes,

several families of exotic fusion systems were constructed; see, for example, [A. Ruiz-Cirera and A. Viruel, *Math. Z.* **248** (2004), no. 1, 45–65; MR2092721].

The only known examples of exotic fusion systems over 2-groups and their associated 2-local finite groups fall into a single family $\mathcal{G}_{\text{Sol}}(q)$, q an odd prime power, and were constructed by R. Levi and R. Oliver [*Geom. Topol.* **6** (2002), 917–990; MR1943386]. This construction has its roots in the work of R. M. Solomon [*J. Algebra* **28** (1974), 182–198; MR0344338], whose results may be interpreted as the fact that $\mathcal{G}_{\text{Sol}}(q)$ is not a 2-local group coming from any finite simple group G with Sylow 2-subgroup S . Levi and Oliver established a relationship between $\mathcal{G}_{\text{Sol}}(q)$ and the exotic 2-adic finite loop space $\text{DI}(4)$ of W. G. Dwyer and C. W. Wilkerson Jr. [*J. Amer. Math. Soc.* **6** (1993), no. 1, 37–64; MR1161306] by showing that the classifying space $B\text{DI}(4)$ is homotopy equivalent to the 2-completion of the nerve of a union of subcategories of certain linking systems $\mathcal{L}_{\text{Sol}}(q^n)$. This result was prefigured and motivated by ideas of D. J. Benson [*Manuscripta Math.* **85** (1994), no. 2, 177–193; MR1302871].

In the present paper, a different construction for the exotic fusion systems $\mathcal{F}_{\text{Sol}}(q)$ is given. This construction is based on the notion of “representation” of a p -local finite group as the p -local group of a not necessarily finite group G , by means of a signalizer functor. It is shown that $\mathcal{F}_{\text{Sol}}(q)$ is the saturated fusion system of a certain free amalgamated product $G = H *_B K$ (described below). Using the signalizer functor, the associated linking system is explicitly constructed.

The paper contains twelve sections. The first three sections are concerned with general principles and supporting results.

In Section 1 known properties and terminology on fusion systems are reviewed. It is worth pointing out that the fusion systems under consideration are defined over p -groups that are not necessarily finite. Correspondingly, the notion of saturation is adapted in such a way that there is no mention of the orders of the normalizers $N_S(P)$ for the subgroups P of S , provided that the fusion system \mathcal{F} is defined over S .

Section 2 reviews the associated centric linking systems. The notion of p -local finite group is extended to the notion of p -local group. A notion of morphisms of p -local groups is also given. One of the most interesting ideas introduced in this section is that of an \mathcal{F} -signalizer functor. Let $\mathcal{F} = \mathcal{F}_S(G)$ be the fusion system associated to a group G . This means that S is a Sylow p -subgroup of G (in the sense that any p -subgroup of G is conjugate to a subgroup of S) and that the fusion system \mathcal{F} may be identified with the fusion system $\mathcal{F}_S(G)$ consisting of all the maps between subgroups of S that are induced by conjugation by elements of G . An \mathcal{F} -signalizer functor is a contravariant functor which assigns to every \mathcal{F} -centric subgroup P of S a group $\theta(P)$ with the property that $C_G(P) = Z(P) \times \theta(P)$. This functor is G -conjugation equivariant and it is inclusion reversing. This definition is inspired by the “balanced signalizer functor” from the theory of finite groups. It is shown that a fusion system \mathcal{F} equipped with a signalizer functor θ gives rise to a centric linking system \mathcal{L}_θ and an associated p -local group $(S, \mathcal{F}, \mathcal{L}_\theta)$.

Amalgams of groups and their associated trees are discussed in Section 3. The authors suggest that the context of amalgams provides a natural framework for the study of fusion systems and their associated p -local groups. This point of view inspired the subsequent papers by I. J. Leary and R. Stancu [*Algebra Number Theory* **1** (2007), no. 1, 17–34; MR2322922] and G. R. Robinson [*J. Algebra* **314** (2007), no. 2, 912–923; MR2344591].

Section 4 provides information on certain spin groups. Let p be an odd prime that is congruent to 3 or 5 mod 8. Let $\overline{\mathbf{F}}$ be an algebraic closure of the field of p elements. Let \mathbf{F} be the subfield of $\overline{\mathbf{F}}$ that is obtained as the union of the tower of subfields of $\overline{\mathbf{F}}$ of order p^{2^n} , $n \geq 0$. Set $\overline{H} \simeq \text{Spin}_7(\overline{\mathbf{F}})$ and let ψ be an endomorphism of \overline{H} with $C_{\overline{H}}(\psi) \simeq \text{Spin}_7(p)$. Then $H = \bigcup_{n \geq 0} C_{\overline{H}}(\psi^{2^n})$. It is shown that all four groups U in H containing $Z(H)$ are conjugate and that the identity component B^0 of $B = N_H(U)$ is a commuting product of three copies of $\text{SL}_2(\mathbf{F})$.

In Section 5 the main ingredients of the proof are introduced, the amalgam $\mathcal{A} = (H \langle B \rangle K)$ and its associated free amalgamated product $G = H *_B K$. Here $K = \langle B, y \rangle$ is a group isomorphic to a split extension of B^0 by the symmetric group on three letters and y is an automorphism of B^0 of order 3, which transitively permutes the three $\text{SL}_2(\mathbf{F})$ components of B . Let T denote a maximal torus in B^0 . Then normalizer $N_H(T)$ contains a Sylow 2-subgroup S of G and in the case when T is y -invariant, the subgroup S is also a Sylow 2-subgroup of each of the groups H , B , and K . Furthermore $S/S \cap T$ may be identified with a Sylow 2-subgroup of $\text{Aut}_G(S/S \cap T)$, a key fact in proving the saturation of the fusion system $\mathcal{F}_S(G)$.

After a succinct interlude on discrete p -toral groups (Section 6) the main objective of the next three sections is to prove saturation of $\mathcal{F}_S(G)$. In Section 7 information on the local subgroups of the free amalgamated product as well as on the fusion among the centric subgroups of S is gathered. Section 8 is dedicated to the ingenious constructions of various signalizer functors. One of the radical centric subgroups of S is an elementary abelian group A of order 16, and there is a surjective homomorphism $\phi_A: N_G(A) \rightarrow L$, with L a certain maximal 2-local subgroup of the sporadic simple group Co_3 . Further, $C_G(A) = A \times \ker(\phi_A)$. Then the union \mathbf{X} of the sets of the form $\ker(\phi_A)^g$ with $g \in G$ is a normal subset of G . Given a centric subgroup P of S , the signalizer functor is defined to be $\theta(P) = C_{\mathbf{X}}(P)O(C_G(P))$.

Section 9 contains the proof of saturation and completes the proof of the first main result, Theorem A. First, it is established that H controls $C_G(Z(H))$ -fusion in S . Next the groups G_σ of fixed points of certain automorphisms σ of G provide representations of the 2-local finite groups considered in [R. Levi and R. Oliver, op. cit.]. In particular, for each such σ , the fusion system $\mathcal{F}_{S_\sigma}(G_\sigma)$ is saturated and a signalizer functor θ_σ is defined, which in its turn gives rise to an associated centric linking system.

The radical centric subgroups of S and S_σ are completely determined in Section 10. Here σ is an automorphism of G that fixes S and which induces a Frobenius endomorphism of H . This information is necessary for the construction of morphisms. A category of p -local groups, having p -local finite groups as a full subcategory, is constructed in Section 11; as an application Theorem B is proven. In the last section limits of directed systems of p -local groups are introduced and Theorem D is proved.

The paper under review is a monumental work and a valuable resource for anyone interested in the theory of fusion systems and their associated categories.

Silvia Onofrei

From MathSciNet, July 2016

MR2752788 (2012f:20063) 20D20; 20D05, 20D10, 20D15

Aschbacher, Michael

The generalized Fitting subsystem of a fusion system.

Memoirs of the American Mathematical Society **209** (2011), no. 986, vi+110 pp.

A fusion system \mathcal{F} over a finite p -group S is a category whose objects are the subgroups of S and whose morphisms are group monomorphisms subject to certain axioms. A fusion system \mathcal{F} is called saturated if it satisfies two more axioms inspired by the Sylow theory in finite groups.

In the memoir under discussion, the author generalizes results from the local theory of finite groups to the setting of saturated fusion systems over finite p -groups. Since it has been observed that theorems on finite groups have easier proofs in the category of fusion systems, one of the main purposes of the memoir's author is proving new theorems about finite groups that are obtained via theorems on fusion systems.

After a short interlude on background and terminology, direct products and intersections of normal fusion subsystems are discussed in Chapters 2 and 3. If \mathcal{F} is a saturated fusion system over S and \mathcal{E}_1 and \mathcal{E}_2 are two normal subsystems, it does not always follow that $\mathcal{E}_1 \cap \mathcal{E}_2$ is saturated. However, there exists a normal subsystem $\mathcal{E}_1 \wedge \mathcal{E}_2$ of \mathcal{F} contained in the intersection. It is worth noting here that the notion of "normal subsystem" used is the one introduced by the author in [Proc. Lond. Math. Soc. (3) **97** (2008), no. 1, 239–271; MR2434097].

The next two chapters, 4 and 5, deal with strongly closed subgroups in \mathcal{F} . Notably, it is shown that the product of two subgroups T_1 and T_2 that are strongly closed in S with respect to \mathcal{F} is strongly closed in S with respect to \mathcal{F} .

Let \mathcal{E} be a normal subsystem of \mathcal{F} . In Chapter 6 the centralizer fusion system $C_{\mathcal{F}}(\mathcal{E})$ is defined and it is proved that it is a normal subsystem of \mathcal{F} . In Chapter 7 various characteristic and subnormal subsystems of \mathcal{F} are analyzed, such as the characteristic subsystem $O^p(\mathcal{F})$ of \mathcal{F} defined on the \mathcal{F} -hyperfocal subgroup of S . Chapter 9 is dedicated to other important standard notions from the theory of finite groups, such as components and generalized Fitting subsystem, which have properties similar to their group theoretic counterparts. In Chapter 10 a theorem of Gorenstein and Walter on L -balance is proved in the context of fusion systems.

In Chapter 11 various properties of the full subcategory \mathcal{F}^c , over the set of all \mathcal{F} -centric subgroups of S , are discussed. Chapter 13 is dedicated to composition factors and a Jordan-Hölder theorem for fusion systems is proved.

In Chapter 14 constrained fusion systems are analyzed. A constrained fusion system can always be realized as the fusion system of a finite group with certain properties. The author defines a fusion system \mathcal{F} to be solvable if all the composition factors of \mathcal{F} are fusion systems of groups of order p . In Chapter 15, it is proved that a saturated fusion system is solvable if and only if \mathcal{F} is constrained and some of the composition factors of the group that realizes \mathcal{F} have prescribed properties.

The last two chapters are dedicated to examples: various fusion systems in simple finite groups are considered in Chapter 16, while Chapter 17 discusses an exotic fusion system due to A. Ruiz.

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Existence and uniqueness of linking systems: Chermak's proof via obstruction theory.

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This paper recasts and extends A. L. Chermak's proof of the existence and uniqueness of centric linking systems for saturated fusion systems [Acta Math. **211** (2013), no. 1, 47-139; MR3118305]. These papers settle the most important open question in the theory of fusion systems, and a question that is, in many ways, responsible for the increasingly vibrant period of research on these categories over the last decade or so.

Let \mathcal{F} be a saturated fusion system over a p -group S , the standard example being given by the fusion system $\mathcal{F}_S(G)$ of a finite group G with Sylow p -subgroup S . The Martino-Priddy conjecture is the statement that two finite groups have equivalent fusion systems at the prime p if and only if the Bousfield-Kan p -completions of their classifying spaces are homotopy equivalent. The reverse direction of this conjecture was proved by J. R. Martino and S. B. Priddy [Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 1, 119-137; MR1356164]. The difficulty in proving the forward direction lies in lifting an automorphism of a fusion system to a homotopy equivalence, a problem whose solution was shown by C. Broto, R. Levi and B. Oliver to be obstructed by a class in $\varprojlim^2(\mathcal{Z}_G)$, where the limit is taken over the p -centric orbit category $\mathcal{O}_p^c(G)$ of G (objects—the p -centric subgroups) and \mathcal{Z}_G is the (contravariant) functor to the category of abelian groups taking a p -centric subgroup to its center. Thus the group $\varprojlim^2(\mathcal{Z}_G)$ obstructs the uniqueness of the canonical p -centric linking system of the group G , whose p -completed nerve is homotopy equivalent with the p -completion of BG [C. Broto, R. Levi and B. Oliver, Invent. Math. **151** (2003), no. 3, 611-664; MR1961340].

For an arbitrary saturated fusion system \mathcal{F} , replace $\mathcal{O}_p^c(G)$ by the orbit category $\mathcal{O}(\mathcal{F}^c)$ of the full subcategory of \mathcal{F} at the \mathcal{F} -centric subgroups. Obstructions to the existence/uniqueness of an associated centric linking system lie in $\varprojlim^3(\mathcal{Z}_{\mathcal{F}})$ and $\varprojlim^2(\mathcal{Z}_{\mathcal{F}})$ [C. Broto, R. Levi and B. Oliver, J. Amer. Math. Soc. **16** (2003), no. 4, 779-856; MR1992826]. The main theorem (Theorem 3.4) of this paper shows that $\varprojlim^i(\mathcal{Z}_{\mathcal{F}}) = 0$ for $i \geq 1$ if p is odd, and for $i \geq 2$ if $p = 2$ and thus that these obstructions vanish. Chermak's proof *a priori* only implied that the specific obstruction in \varprojlim^3 vanishes and that \varprojlim^2 vanishes.

As consequences, a new proof of the Martino-Priddy conjecture is obtained, one can regard the p -completed nerve of the unique linking system as a classifying space for \mathcal{F} (Theorem B), and one obtains a description (Theorem C) of the group of homotopy classes of self-homotopy equivalences of this classifying space. Like the author's original proof of the Martino-Priddy conjecture, both Chermak's proof and the version of it appearing here depend on the Classification of Finite Simple Groups. However, in the latter two cases, the dependence is indirect and localized to one statement about vanishing of \varprojlim^1 and \varprojlim^2 of certain functors when \mathcal{F} is the fusion system of a p -constrained finite group (Proposition 3.2).

The heart of Chermak's proof gives an ingenious way, using the properties of the Thompson subgroup-functor J , to obtain a very useful filtration of the collection of \mathcal{F} -centric subgroups of the fusion system. The linking system is then constructed

iteratively on larger collections of this filtration, starting with the overgroups in S of $J(S)$, where a linking system exists by the Model Theorem of Broto-Castellana-Grodal-Levi-Oliver. From the point of view of this paper, Chermak's filtration gives a way to filter $\mathcal{Z}_{\mathcal{F}}$ by subfunctors in such a way that the relevant limits of each subquotient can be shown to vanish. Within the induction, one is working locally in the normalizer fusion system $N_{\mathcal{F}}(X)$ of a subgroup $X \leq S$. This is a fusion system of a finite group Γ containing $N_S(X)$ as a Sylow p -subgroup and such that $C_{\Gamma}(X) \leq X$ (again by the Model Theorem). It then suffices to show that the limits of a certain subquotient functor $\mathcal{Z}_{\Gamma}^{\mathcal{R}}$ of \mathcal{Z}_{Γ} vanish. Long exact sequences reduce the problem to showing that \varprojlim^1 (resp. \varprojlim^2) vanishes for p odd (resp. $p = 2$) for these functors. The properties of the filtration further allow a reduction to the case that $G = \Gamma/C_{\Gamma}(Z(X))$ is generated by quadratic best offenders on $Z(X)$ and that $O_p(G) = 1$. Now the Meierfrankfeld-Stellmacher classification of such groups (the source of the reliance on the CFSG) gives a list of the possibilities for G [U. Meierfrankfeld and B. Stellmacher, *J. Algebra* **351** (2012), 1–63; MR2862198].

It should be noted that in treating the residual cases, the arguments here are very different from those in Chermak's proof and use methods established by Jackowski-McClure-Oliver, Grodal, and Oliver for computing limits of "atomic" functors, i.e. those functors which are zero except on a single conjugacy class of p -subgroups.

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