BOOK REVIEWS

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Hangzhou lectures on eigenfunctions of the Laplacian, by Christopher D. Sogge, Annals of Mathematics Studies, Vol. 188, Princeton University Press, 2014, xii+193 pp., ISBN 978-0-691-16078-8, US $75.00

1. Introduction

The study of eigenvalues and eigenfunctions of the Laplacian on a compact Riemannian manifold (often called *spectral geometry*) is an attractive combination of several branches of mathematics. First and foremost, it involves partial differential equations and operator theory. But it also touches on Fourier analysis, quantum theory, differential geometry, dynamical systems, ergodic theory, number theory, hyperbolic geometry, Lie groups, index theory, probability, and other fields.

One could say that this topic started with Fourier analysis, specifically Fourier series, that is, the representation of functions on the torus, $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ as superpositions of the functions $e^{2\pi i x\cdot k}$, where k has integer coefficients. From the point of view of Riemannian manifolds, the torus $\mathbb{T}^d$ with its flat metric is just a specific example of a closed Riemannian manifold. Let M be a closed Riemannian manifold, and let g be a Riemannian metric on M, that is, an inner product on each tangent space T_pM, varying smoothly with p. On any such (M, g), the Dirichlet form can be defined. This form maps suitable functions $u : M \to \mathbb{C}$ to the integral of the square of their gradient:

$$Q(u) = \int_M |\nabla u(x)|^2 \, d\mathrm{vol}.$$

On the right-hand side of this formula, the metric g is used three times. First, g determines an isomorphism between T_pM and T_p^*M for each p, and using this, we define the gradient ∇u as the inverse image of the differential du under this isomorphism. Next, we use the inner product on T_pM to take the squared norm of $\nabla u(p)$, giving us a function on M. Finally, g determines a canonical measure $d$vol on M, with respect to which we integrate this function, to obtain $Q(u)$.

The quadratic form Q, acting on the Sobolev space[1] $H^1(M)$, turns out to be closed,[2] and hence is associated with a self-adjoint operator on $L^2(M)$ (with do-

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[1]The Sobolev space $H^1(M)$ is the Hilbert space of functions u such that $|\nabla u| \in L^2(M)$, with squared norm $\|u\|^2_{L^2(M)} + Q(u)$.

[2]This is a technical condition guaranteeing that a quadratic form is associated to an operator; see any functional analysis text for the definition.

main $H^2(M)$), which one calls minus[3] the Laplacian on M, and denotes $-\Delta$, or $-\Delta_g$ if one wishes to emphasize the dependence on g. Since Q is nonnegative, $\mathrm{Id}-\Delta$ is a strictly positive operator, hence invertible. Moreover, $-\Delta$ is an *elliptic* operator; elliptic estimates, together with Rellich's compactness lemma shows that $(\mathrm{Id}-\Delta)^{-1}$ is compact. This implies that $-\Delta$ has pure-point spectrum, that is, the spectrum consists of a sequence of eigenvalues E_j tending to $+\infty$, with associated finite-dimensional eigenspaces, which are (in view of the self-adjointness of $-\Delta$) mutually orthogonal and complete. In particular, there are orthonormal bases $L^2(M)$ consisting of eigenfunctions u_j of $-\Delta$, unique up to orthogonal transformations in each eigenspace. That is, one can expand suitable functions (L^2 functions, say) in eigenfunctions of the Laplacian. Since the Fourier modes $e^{2\pi i x\cdot k}$, $k \in \mathbb{Z}^d$, are the eigenfunctions of the Laplacian on the torus $\mathbb{T}^d$, Fourier series are just one example of this general phenomenon.

The flat torus is, of course, an exceedingly special Riemannian manifold geometrically (it has vanishing curvature), algebraically (it is an abelian Lie group, hence has a large isometry group), dynamically (its geodesic flow is completely integrable), and number-theoretically (its dual group is an integral lattice). One can ask: to what extent are properties of Fourier series shared by eigenfunction expansions on any closed Riemannian manifold, and to what extent are Fourier expansions genuinely special? More precisely, how do the geometric, algebraic, dynamical, and number-theoretic properties of Riemannian manifolds affect properties of eigenfunction expansions? Much recent research on eigenfunctions of the Laplacian can be viewed as an answer to this question.

2. Short time analysis and Weyl asymptotics

The study of eigendata of the Laplacian on a Riemannian manifold splits into two strands: the study of low eigendata (the first and second eigenvalues, say, and corresponding eigenfunctions) or high eigendata (the asymptotic properties of eigendata as the eigenvalue tends to infinity). These two strands are surprisingly distant from each other, both in the flavour of results and in the techniques used. The book under consideration, and this review, is only concerned with the second strand. Let us first discuss results concerning asymptotic distribution of eigenvalues. These are encoded in the asymptotics of the counting function $N(\lambda)$ as $\lambda \to \infty$, where $N(\lambda)$ is defined as the number of eigenvalues, counted with multiplicity, less than λ^2.[4] On the torus $\mathbb{T}^d$, eigenvalues of the Laplacian take the form $(2\pi)^2|k|^2$, where k varies over $\mathbb{Z}^d$. Therefore, for the d-dimensional torus, $N(\lambda)$ is precisely the number of points in $\mathbb{Z}^d$ contained in the open ball of radius $\lambda/(2\pi)$. It is easy to see that this has a leading asymptotic which is the volume of this ball, that is, $(\lambda/2\pi)^d\omega_d$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$. It is a classical problem in number theory, known as the Gauss circle problem, to determine how precise this asymptotic is, that is, to estimate the remainder term $R(\lambda) := N(\lambda) - (2\pi\lambda)^d\omega_d$. Gauss was able to show that $R(\lambda) = O(\lambda)$ in dimension $d = 2$. Using more

[3]The minus sign here is included for historical reasons. It is an inconvenience from the spectral theory point of view, and some authors (including this reviewer) like to eliminate it. However, most authors, including the author of the book under review, retain the minus sign, and we shall follow this convention here.

[4]We use λ^2, rather than λ, to denote the eigenvalue, following the book under review. Many articles use λ rather than λ^2 to denote the eigenvalue; both are common conventions.

elaborate, number-theoretic ideas, this has been improved to $O(\lambda^\alpha)$ for $\alpha > 131/208$ [Hux].

What about for general Riemannian manifolds? What one can think of as the *fundamental theorem of spectral geometry* is the Weyl asymptotic formula, which states that, on any Riemannian manifold (M, g), we have a leading asymptotic

$$N(\lambda) \sim \big(\frac{\lambda}{2\pi}\big)^d \omega_d \operatorname{vol}(M, g). \tag{2.1}$$

In other words, the remainder,

$$R(\lambda) := N(\lambda) - \big(\frac{\lambda}{2\pi}\big)^d \omega_d \operatorname{vol}(M, g) \quad \text{ is } o(\lambda^d) \text{ as } \lambda \to \infty. \tag{2.2}$$

The proof of this asymptotic, together with the remainder estimate

$$R(\lambda) = O(\lambda^{d-1}), \quad \lambda \to \infty, \tag{2.3}$$

which is sharp in the sense that the exponent $d-1$ cannot be improved on a general manifold, occupies Chapters 2 and 3 of the book. It is such a beautiful combination of PDEs and operator theory that we cannot resist discussing it here in some detail.

We start by observing that $N(\lambda)$ is precisely the trace of the spectral projector $E(\lambda)$, the projection onto the direct sum of eigenspaces with eigenvalue less than λ^2. The operator $E(\lambda)$ is a function of the Laplacian $-\Delta$, namely $f(-\Delta)$, where f is the characteristic function of $[0, \lambda^2)$, but it is not a very convenient function to study directly. A key idea is to study *functions of $-\Delta$, depending on a time parameter t, which satisfy an evolution equation in time*—this allows us to exploit PDE theory. For example, we can use the exponential decay function $f_t(\sigma) = e^{-t\sigma}$, which we can think of as a smooth version of the characteristic function of $[0, \lambda^2)$, if $t = \lambda^{-2}$. The operator $f_t(-\Delta)$, which we naturally denote $e^{t\Delta}$, satisfies the evolution equation

$$\big(\frac{\partial}{\partial t} - \Delta\big)e^{t\Delta} = 0. \tag{2.4}$$

PDE theory shows that $p_t(x, y)$, the Schwartz kernel[5] of $e^{t\Delta}$ (the *heat kernel*), is C^∞ for $t > 0$, hence $e^{t\Delta}$ is of trace class for $t > 0$. This implies that

$$\operatorname{tr} e^{t\Delta} = \int_M p_t(x, x) d\mathrm{vol}(x) = \sum_j e^{-tE_j},$$

where this sum is over all eigenvalues E_j, repeated according to their multiplicity, and converges for $t > 0$. Moreover, detailed PDE analysis of $p_t(x, y)$ near $\{t = 0, x = y\}$ shows that $p_t(x, y)$ has a Gaussian profile here, concentrated at the diagonal,[6] with variance about $\sqrt{2t}$ from which we deduce that

$$\int_M p_t(x, x) d\mathrm{vol}(x) \sim (4\pi t)^{-n/2} \operatorname{vol}(M), \ t \to 0.$$

Finally, the so-called Karamata Tauberian theorem asserts that if $\sum_j e^{-tE_j}$ has a leading asymptotic $(4\pi t)^{-n/2} \operatorname{vol}(M)$ as $t \to 0$, then $N(\lambda) = \sum_j 1_{[0,\lambda^2)}(E_j)$ has a leading asymptotic as in (2.2).

[5] For algebraically minded readers, we emphasize that here, "kernel" is always used in the sense of "integral kernel", not "null space".

[6] For example, on $\mathbb{R}^d$, with the flat metric, $p_t(x, y) = (4\pi t)^{-d/2} e^{-|x-y|^2/4t}$ is precisely Gaussian.

However, this is far from the sharp estimate (2.3), and it seems impossible to obtain it, or even to improve upon (2.2), using the heat kernel. The problem is that in the trace of the heat kernel $\sum_j e^{-tE_j}$, the large eigenvalues are exponentially suppressed. This suppression is advantageous from some points of view—for example, it is responsible for the heat operator $e^{t\Delta}$ being in the trace class—but precisely because it suppresses large eigenvalues, it is an inefficient tool for deducing their asymptotic behaviour. An alternative, introduced by Hörmander [Hör], is $\cos(t\sqrt{-\Delta})$, which also satisfies a PDE, this time the wave equation:

$$(\frac{\partial^2}{\partial t^2} - \Delta) \cos(t\sqrt{-\Delta}) = 0. \tag{2.5}$$

The corresponding trace is, formally, $\sum_j \cos(t\sqrt{E_j})$, which certainly does not suppress the contribution of large eigenvalues, but it appears to have the fatal disadvantage that it fails to converge. However, it is an astonishing fact that this sum makes sense distributionally and even converges to a smooth function on any open t-interval that is disjoint from the lengths of periodic geodesics on M! To see why, one must return to PDE theory and construct the Schwartz kernel of the operator $\cos(t\sqrt{-\Delta})$. For $t > 0$, it turns out to be the derivative in time of the fundamental solution[7] for the PDE initial value problem

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - \Delta)u(x,t) &= 0, \\ u(x,0) &= 0, \\ \frac{\partial}{\partial t}u(x,0) &= f(x). \end{aligned} \tag{2.6}$$

As for the heat equation, there is an asymptotic expansion for the fundamental solution, valid for small time, known as the Hadamard parametrix. Unlike the heat equation, it is singular for positive time. However, the singularities can be specified very precisely. The Hadamard parametrix is a sum of terms, each of which is singular at the *light cone*, that is, the set $\{(x,y) \mid d_g(x,y) = t\}$, where d_g is the Riemannian distance function and is a homogeneous function (or distribution) of $t - d_g(x,y)$. Moreover, the exact solution to (2.6) is equal to the Hadamard parametrix up to a smooth error. Using this precise characterization of the singularities of the forward fundamental solution of (2.6), it is not hard to show the following: for any Schwartz function $\rho(t)$ of t, the Schwartz kernel of the integral

$$\int \cos(t\sqrt{-\Delta})\rho(t)\,dt \tag{2.7}$$

is smooth, and hence (2.7) is a trace class operator. That is, the trace of $\cos(t\sqrt{-\Delta})$ is meaningful as a tempered distribution in t. Moreover, one can show that when the singularities of the Schwartz kernel of $\cos(t_*\sqrt{-\Delta})$ do not meet the diagonal, then $\operatorname{tr}\cos(t\sqrt{-\Delta})$ is actually smooth near $t = t_*$, and the trace is given by the integral of the restriction of the Schwartz kernel to the diagonal. In particular, in a small time interval containing zero, the trace of $\cos(t\sqrt{-\Delta})$ is singular only at $t = 0$. Close examination of the Hadamard parametrix shows that the trace is *conormal* at $t = 0$, that is, if $\rho \in C_c^\infty(R)$ has sufficiently small support around 0, then the Fourier transform of $2\rho(t)\operatorname{tr}\cos(t_*\sqrt{-\Delta})$ has an expansion in powers of the dual variable λ, with principal term $\sim \lambda^{d-1}$. This is, however, the trace of the

[7]The fundamental solution to (2.6) is the distribution $w_t(x,y)$ that solves (2.6) with $f = \delta(y)$.

operator $\hat\rho(\sqrt{-\Delta} - \lambda) + \hat\rho(\sqrt{-\Delta} + \lambda)$. Since $\hat\rho$ is rapidly decreasing, the second operator is easily shown to be rapidly decreasing in λ, and we find that the trace of $\hat\rho(\sqrt{-\Delta} - \lambda)$ has an expansion $\sim \lambda^{d-1}$ as $\lambda \to \infty$. By integrating this, we find that $N(\lambda) * \hat\rho$ has an expansion $(\lambda/2\pi)^d \omega_d \operatorname{vol}(M) + O(\lambda^{d-1})$. From there, a simple but clever argument yields (2.3).

It is worth remarking that, although the discussion above appears to concern only the eigenvalues and not the eigenfunctions, that is not actually the case. This is because the statement that $N(\lambda) = (\lambda/2\pi)^d \omega_d \operatorname{vol}(M) + O(\lambda^{d-1})$ can be refined to a pointwise statement. Indeed, $N(\lambda)$ is the trace of $E(\lambda)$, and is therefore given by the integral of the Schwartz kernel of $E(\lambda)$ on the diagonal. If we denote this Schwartz kernel by $E(\lambda)(x, y)$, we thus have

$$N(\lambda) = \int E(\lambda)(x, x) \, d\mathrm{vol}.$$

One can actually prove the pointwise statement that

$$E(\lambda)(x, x) = \big(\frac{\lambda}{2\pi}\big)^d \omega_d + O(\lambda^{d-1}), \qquad \text{for each } x \text{ in } M.$$

This means that

$$E(\lambda + 1)(x, x) - E(\lambda)(x, x) = O(\lambda^{d-1}).$$

But, writing the spectral projection in terms of the normalized eigenfunctions u_j, for $E_j < \lambda^2$, implies that

$$\sum_{\lambda^2 \le E_j < (\lambda+1)^2} |u_j(x)|^2 = O(\lambda^{d-1})$$

uniformly in x. This implies in particular that the L^∞ norm of any (L^2-normalized) eigenfunction with eigenvalue λ^2 is $O(\lambda^{(d-1)/2})$. Moreover, improvements in the remainder estimate automatically lead to improved L^∞ estimates for the u_j.

3. Long time analysis

The discussion above roughly covers the same ground as the first half of the book under review: the sharp Weyl estimate is proved, and is shown to be sharp for the sphere S^d with its round metric. In the second half (Section 3.5 onward), Sogge discusses situations where one can go beyond this universal estimate. The flat torus, and negatively curved manifolds, are two cases of particular geometries where the remainder estimate can be improved. The improvement in the case of the torus is particularly satisfying, as the estimate is improved by a whole power: for example, it becomes $O(\lambda^{2/3})$ instead of $O(\lambda)$ for $\mathbb{T}^2$. The argument, as presented by Sogge, uses the wave equation and a little Fourier analysis—it is rather gratifying to see the efficacy of PDE theory in tackling Gauss's circle problem! The improvement in the case of negatively curved manifolds is more subtle—the remainder is shown to be $O(\lambda^{d-1}/\log\lambda)$, only a logarithmic improvement over (2.3). Both of these improvements stem from the fact that the Riemannian manifolds in question have a universal cover which has no conjugate points, and the wave kernel can then be estimated for large time. This overcomes the constraint that the uncertainty principle places on our remainder estimate. In fact, if our analysis of the wave kernel is local in time, say, restricted to a fixed interval containing $t = 0$, then the uncertainty principle tells us that we cannot localize in the dual variable to time to finer than an $O(1)$ scale. The dual variable is frequency, denoted λ in our notation above, hence we cannot localize λ to finer than $O(1)$. Since our leading asymptotic is $c\lambda^d$, it follows that $N(\lambda)$ cannot be determined to better than $(\lambda + O(1))^d - \lambda^d$, that is, to $O(\lambda^{d-1})$. Thus our ability to "beat" the $O(\lambda^{d-1})$ barrier depends on effective estimates on the fundamental solution of (2.6) for times larger than $O(1)$.

Additional microlocal machinery (introduced in Chapter 4 of these lectures) allows us to analyze the fundamental solution of (2.6) for long time, and thereby to go beyond the results obtained in the first half of the book, under *dynamical* assumptions on the manifold (i.e., the long time behaviour of the geodesic flow), as opposed to the *geometric* assumptions in the previous paragraph (the key property being nonpositive curvature). Two such results are presented in the second half of the book. The first is the *Duistermaat–Guillemin Theorem* [DG], which states that, if the union of all periodic geodesics in T^*M has zero measure, then the remainder term $R(\lambda)$ is not just $O(\lambda^{d-1})$, but $o(\lambda^{d-1})$. The key to proving this is a local version of the Weyl law. The Weyl law can be written

$$\sum_{E_j<\lambda^2} \|u_j\|^2 \sim \big(\frac{\lambda}{2\pi}\big)^d \omega_d \operatorname{vol}(M), \tag{3.1}$$

since each eigenfunction u_j is normalized so that $\|u_j\| = 1$. The sum in (3.1) can be localized by introducing a pseudo-differential operator A (think of an operator whose symbol $\sigma(A)$ is supported in a small conic set in $T^*X \setminus 0$). The local Weyl law says that

$$\sum_{E_j<\lambda^2} \|Au_j\|^2 \sim \big(\frac{\lambda}{2\pi}\big)^d \int_{|\xi|_g\le\lambda} \sigma(A)(x,\xi)\, dxd\xi, \tag{3.2}$$

which reduces to (3.1) when A is the identity, since then its symbol is identically 1. Using the homogeneity of the principal symbol of A and (2.1), (3.2) can be written equivalently as

$$\lim_{j\to\infty} \frac{1}{N(\lambda)} \sum_{E_j<\lambda^2} \|Au_j\|^2 = (2\pi)^{-d} \int_{S^*M} \sigma(A)(x,\xi). \tag{3.3}$$

Using this, for any time $T > 0$, we can divide the phase space T^*M into two open sets, a *good set*, where the geodesic flow is nonperiodic up until time T, and a *bad set*, which includes all the periodic points with period $\le T$, but has small measure. Then, very roughly, using two pseudo-differential operators to localize to the good and bad sets, the bad set contributes only a small constant (proportional to its measure) to the remainder estimate using (3.2), while the good set contributes a term proportional to $1/T$ (as permitted by the uncertainty principle), which is small if we take T sufficiently large. This theorem is discussed in detail in Chapter 5.

The second key result that exploits long-time behaviour of the wave kernel is the *Quantum Ergodicity Theorem* of Schnirelman, Zelditch, and Colin de Verdière [Sch, Zel, CdV]. Like the Duistermaat–Guillemin theorem, it makes a global dynamical assumption on the geodesic flow. This theorem says that, if the classical flow is ergodic, then the sequence of eigenfunctions is *quantum ergodic*, which means that a density one subsequence is equidistributed in the following sense: along this subsequence, we have

$$\lim_{\lambda\to\infty} \|Au_j\|^2 = (2\pi)^{-d} \int_{S^*M} \sigma(A)(x,\xi). \tag{3.4}$$

That is, the limit (3.2) actually holds pointwise along a density one subsequence of eigenfunctions, without averaging.

4. Hangzhou lectures

We will conclude with a few general remarks about the book under review. The book is derived from lectures delivered by the author at Zhejiang University in Hangzhou, China. These lectures assume remarkably little in the way of background knowledge. The only things assumed are basic real analysis (Lebesgue integration, the basics of Fourier analysis, and some simple properties of Hilbert and L^p spaces), a little distribution theory (but even this is covered in an appendix), and some familiarity with manifolds. Hardly anything in the way of PDE theory is assumed (even elliptic estimates are covered in Chapter 1), and no Riemannian geometry is required. Nevertheless, the book covers ground fast, and it rapidly proves several foundational results in the study of Laplace eigenfunctions, with the author's customary efficiency. The book will be invaluable for graduate students learning this topic.

One cannot expect a book of two-hundred-odd pages to cover this topic comprehensively, and the author has inevitably had to leave out several major topics, even omitting his own celebrated L^p estimates on eigenfunctions [Sog] (though it is worth mentioning that these are covered in Sogge's earlier book [Sog2], for which this current book well serves as an introduction). Manifolds with boundary are not discussed at all, for example, nor is there a discussion of recent research directions or open problems. Nonetheless, these lecture notes are a concise, elegant and valuable addition to the literature on eigenfunctions of the Laplacian on Riemannian manifolds.

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