TOPOLOGICAL MANIFOLDS AND SMOOTH(1) MANIFOLDS

By J. MILNOR

Suppose that one is given a topological manifold M (i.e. a Hausdorff space with a countable basis where each point has a neighborhood homeomorphic to some euclidean space). Then one can ask the following two questions:

Problem 1. Can M be given the structure of a smooth manifold? In more intuitive terms: can M be imbedded in a high dimensional euclidean space so as to have a continuously turning tangent plane?

Problem 2. If such a smoothness structure exists, is it essentially unique? More precisely, given two such structures on M, does there exist a homeomorphism of M onto itself which carries one structure to the other?(2)

The first problem was answered negatively when M. Kervaire gave an example of a compact triangulable 10-dimensional manifold which is not smoothable. (Thus if Kervaire's manifold is imbedded in some euclidean space, its image must have "angles" or "corners" or worse singularities.) Other such examples, in other dimensions, have been given by Smale, Tamura, Wall and by Eells and Kuiper (references [5], [6], [14], [18], [21]).

The second problem was answered negatively when the author showed that the 7-dimensional sphere possesses several essentially distinct smoothness structures (see [8], [9], [13], [17]).

Thus the two problems are non-trivial. They lead naturally to the following.

Problem 3. Given a topological manifold M, can one make a classification of all possible smoothness structures on M?

The answer must surely depend on a detailed knowledge of the topology of M.

Quite a bit of progress on these questions has been made during the last few years. Suppose for example that M is the topological sphere S^n. Define two smoothness structures on S^n to be *equivalent* if there exists an orientation preserving homeomorphism of S^n to itself which carries one smoothness structure to the other. For $n \neq 4$ it is known that the set of such equivalence classes can be made into an abelian group, which is denoted by Γ_n. The structure of this group for many small values of n has been determined by Kervaire and Milnor, making use of work by Smale (references [7], [10], [14], [15]). (For the cases $n < 4$ see [11], 22].) For example one has:

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_5 = \Gamma_6 = 0, \quad \Gamma_7 = Z_{28}, \quad \Gamma_8 = Z_2.$$

The groups Γ_n, $n \neq 4$, are all finite.(3)

(1) The word *smooth* will be used as a synonym for "differentiable of class C^∞".

(2) This is equivalent to the question as to whether the two resulting smooth manifolds are diffeomorphic to each other.

(3) Here Z_k denotes the cyclic group of order k. For $n = 4$ the group Γ_4 must be defined somewhat differently. Nothing is known about the structure of Γ_4.

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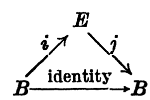
Now let M be an arbitrary triangulated manifold. J. Munkres, in reference [12], has defined a sequence of obstructions, whose vanishing implies that M can be given a smoothness structure (see also Thom [19], [20]). These obstructions are homology classes of M with coefficients in the groups Γ_i. Similarly, if one is given two different smooth manifolds with the same underlying complex, Munkres [11] has defined a sequence of obstruction classes whose vanishing implies that the two manifolds are diffeomorphic. Again the groups Γ_i occur as coefficient groups.

One interesting application of Munkres' results has been made by J. Stallings. In reference [16], Stallings shows that the euclidean space $R^n, n \neq 4$, has an essentially unique smoothness structure.

In the remainder of this lecture, I would like to introduce a quite different tool, which I hope will be used in the future to attack these problems; namely the theory of microbundles.

A "microbundle" is an object something like a fibre bundle having the euclidean space R^n as fibre. However the fibre in a microbundle is not an honest topological space, but is only a "germ" of a topological space. This can be made precise as follows.

Definition. An *R^n-microbundle* over B is a commutative diagram

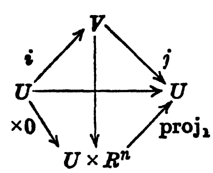


where B, E are topological spaces, and i, j are continuous maps; such that the following "local triviality" condition is satisfied:

Requirement. For each $b \in B$ there should exist neighborhoods U of b and V of $i(b)$, with

$$i(U) \subset V, \; j(V) = U$$

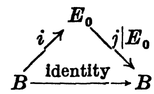
so that V is homeomorphic to $U \times R^n$ under a homemorphism which makes the following diagram commutative



Here R^n denotes the n-dimensional euclidean space, $\times 0$ denotes the mapping $u \rightarrow (u, 0)$, and proj_1 denotes the projection to the first factor: $\text{proj}_1\,(u, x) = u$.

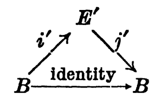
Such a microbundle will be denoted by a single German letter, such as $\mathfrak{x}$. The spaces E and B will be called the *total space* and the *base space* respectively. The maps i, j will be called the *injection* and the *projection* maps of $\mathfrak{x}$.

Note that this condition of local triviality depends only on that portion of E which lies in a aribitrarily small neighborhood of $i(B)$. If E_0 is any neighborhood of $i(B)$ in E then we will take the point of view that the new microbundle

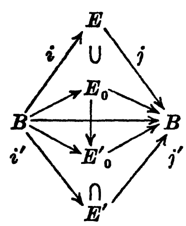


can be identified with the original one. More precisely, and more generally:

Definition. A second microbundle $\mathfrak{x}'$ over B with diagram

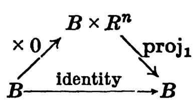


is *isomorphic* to $\mathfrak{x}$ if there exist neighborhoods E_0 of $i(B)$ in E and E_0' of $i'(B)$ in E', and a homeorphism from E_0 to E_0' which makes the following diagram commutative.



Here are some examples of microbundles.

Example 1. For any topological space B and any integer $n \geqslant 0$ one has the *trivial* microbundle $\mathfrak{e}^n$ with diagram

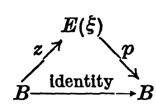


More generally any microbundle isomorphic to $\mathfrak{e}^n$ is called a trivial microbundle.

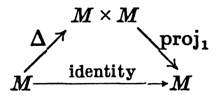
Example 2. Let ξ be a vector bundle over B with total space $E(\xi)$ and projection map $p: E(\xi) \to B$. There is a standard cross-section

$$z: B \to E(\xi)$$

which assigns to each $b \in B$ the zero vector in the vector space $p^{-1}(b)$. The *underlying microbundle* $|\xi|$ of ξ is defined to be the microbundle



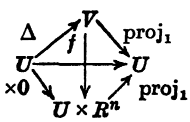
Example 3. Let M be a topological manifold. Then the *tangent microbundle* $\mathfrak{t}$ of M is defined to be the microbundle



where Δ denotes the diagonal map. Thus the "fibre" over a point $x_0 \in M$ is the set of all pairs (x_0, y) where y ranges over an arbitrary neighborhood of x_0 in M. The local triviality condition can be verified as follows. Given $x_0 \in M$ let U be a neighborhood homeomorphic to R^n under a homeomorphism h, and let $V = U \times U$. Then V is homeomorphic to $U \times R^n$ under the homeomorphism,

$$f(u_1, u_2) = (u_1, h(u_2) - h(u_1)),$$

which makes the following diagram commutative.



Now suppose that M can be made into a smooth manifold. Then, using the smoothness structure, one can also define the *tangent vector bundle* τ of M. The following result is fundamental.

THEOREM 1. *In this situation the underlying microbundle* $|\tau|$ *is isomorphic to the tangent microbundle* $\mathfrak{t}$ *of* M.

The proof can be outlined as follows. Choose a Riemannian metric on M. Then for any tangent vector $v \in E(\tau)$ which is not too long, there exists a geodesic segment

$$\gamma_v : [0, 1] \rightarrow M,$$

whose velocity vector at 0 is the given vector v. Now the correspondence

$$v \rightarrow (\gamma_v(0), \gamma_v(1))$$

defines the required homeomorphism between a neighborhood of $z(M)$ in $E(\tau)$ and a neighborhood of the diagonal in $M \times M$.

COROLLARY. *If* M *can be smoothed then the tangent microbundle* $\mathfrak{t}$ *is isomorphic to* $|\xi|$ *for some vector bundle* ξ *over* M.

A fundamental conjecture would be the converse proposition:

Problem. If $\mathfrak{t}$ is isomorphic to $|\xi|$ for some ξ, does it follow that M can be given a smoothness structure?

The following partial result can be proved.

THEOREM 2. *If the tangent microbundle of* M *is isomorphic to* $|\xi|$ *for some vector bundle* ξ*, then the Cartesian product* $M \times R^{4n+1}$ *can be given a smoothness structure.*

I will not try to describe the proof, which is based on a method due to M. Curtis and R. Lashof [4].

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Many standard constructions for vector bundles carry over immediately to microbundles. For example given a microbundle $\mathfrak{x}$ over B, and given a map $f: B' \to B$, one can construct the *induced microbundle* $f^*\mathfrak{x}$ over B'.

THEOREM 3. (Homotopy theorem.) *If B' is paracompact, and if $g: B' \to B$ is homotopic to f, then the induced bundle $g^*\mathfrak{x}$ is isomorphic to $f^*\mathfrak{x}$.*

The proof is similar to the usual proof for vector bundles.

Given two microbundles $\mathfrak{x}$ and $\mathfrak{y}$ over the same base space B, one can construct the *Whitney sum* $\mathfrak{x} \oplus \mathfrak{y}$, a new vector bundle over B. By definition, $\mathfrak{x} \oplus \mathfrak{y}$, is equal to $\Delta^*(\mathfrak{x} \times \mathfrak{y})$, where $\mathfrak{x} \times \mathfrak{y}$ denotes the Cartesian product microbundle

$$\begin{array}{ccccc} & & E(\mathfrak{x}) \times E(\mathfrak{y}) & & \\ & {\scriptstyle i \times i'} \nearrow & & \searrow {\scriptstyle j \times j'} & \\ B \times B & & \xrightarrow{\text{identity}} & & B \times B \end{array}$$

and where $\Delta: B \to B \times B$ denotes the diagonal map.

Definition. Two microbundles $\mathfrak{x}$ and $\mathfrak{x}'$ over B belong to the same s-class if there exist integers m, m' so that the Whitney sum $\mathfrak{x} \oplus \mathfrak{e}^m$ is isomorphic to $\mathfrak{x}' \oplus \mathfrak{e}^{m'}$. (Here $\mathfrak{e}^m$ denotes the trivial R^m-microbundle over B.)

THEOREM 4. *Let B be a finite dimensional complex. Then the s-classes of microbundles over B form an abelian group with respect to the Whitney sum operation.*

The proof is more difficult than the corresponding proof for vector bundles. The key step, showing that for each $\mathfrak{x}$ there exists a $\mathfrak{y}$ with $\mathfrak{x} \oplus \mathfrak{y}$ trivial, is proved by induction on the dimension of B.

This group of s-classes of microbundles will be denoted by $k_{\text{Top}}B$. The analogous group whose elements are s-classes of vector bundles over B will be denoted by $k_{\text{Orthog}}B$. Note that the correspondence $\xi \to |\xi|$ gives rise to a natural homomorphism

$$k_{\text{Orthog}}B \to k_{\text{Top}}B.$$

Note also that the groups $k_{\text{Top}}\ B$ (or $k_{\text{Orthog}}B$) behave somewhat like cohomology groups. Thus any map $f: B' \to B$ induces a homomorphism

$$f^*: k_{\text{Top}}B \to k_{\text{Top}}B'.$$

If f is a homotopy equivalence, then f^* is an isomorphism.

The groups $k_{\text{Orthog}}B$ are well known through the work of Atiyah, Hirzebruch, Adams and others (see [1], [2], [3]). Unfortunately very little is known about $k_{\text{Top}}B$. For example it is not known whether the groups $k_{\text{Top}}S^n$ are finite, countably infinite, or uncountably infinite. Even the group $k_{\text{Top}}S^1$ seems forbiddingly difficult to compute.

The following qualitative result can be obtained.

THEOREM 5. *There exists a finite complex X_1 for which the canonical homomorphism*

$$k_{\text{Orthog}}X_1 \to k_{\text{Top}}X_1$$

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has a non-trivial kernel. Furthermore there exists a finite complex X_2 so that the canonical homomorphism

$$k_{\mathrm{Orthog}} X_2 \to k_{\mathrm{Top}} X_2$$

is not onto.

Thus the theory of microbundles is essentially distinct from the theory of vector bundles. The proof of Theorem 5 is quite difficult. It is based on joint research with M. Kervaire [7].

(Actually the proof of Theorem 5 gives a specific example of such a complex X_1: namely a 7-sphere with an 8-cell attached by a map of degree 7. For X_2 the proof shows only that one of two possibilities will work. If $k_{\mathrm{Top}} S^8$ is infinite then the 8-sphere itself will serve as a complex X_2. If $k_{\mathrm{Top}} S^8$ is finite, then the 8-sphere with a 9-cell attached by a map of degree 3 will serve.)

Each half of Theorem 5 has an interesting consequence.

COROLLARY 1. *The tangent vector bundle of a certain smooth manifold M_1 is not a topological invariant.*

Proof. Choose an open set U_1 in some euclidean space R^n which has the same homotopy type as X_1. Then there exists a vector bundle ξ over U_1 whose s-class is non-trivial, and belongs to the kernel of the homomorphism $k_{\mathrm{Orthog}} U_1 \to k_{\mathrm{Top}} U_1$. Thus the underlying microbundle $|\xi|$ is s-trivial. Without loss of generality we may assume that $|\xi|$ itself is trivial.

Let ε^p denote the trivial vector bundle, with total space $U_1 \times R^p$, where p is the fibre dimension of ξ. Since $|\varepsilon^p|$ is isomorphic to $|\xi|$ it follows that some neighborhood M_1 of $U_1 \times 0$ in $U_1 \times R^p$ is homeomorphic to some neighborhood M_1' of the zero cross-section in $E(\xi)$.

But each of the bundles ε^p and ξ can be given the structure of a smooth vector bundle. Hence the open sets $M_1 \subset E(\varepsilon^p)$ and $M_1' \subset E(\xi)$ can be considered as smooth manifolds. Clearly the manifold M_1 is parallelizable. However the tangent vector bundle of M_1', restricted to U_1, is isomorphic to

$$(\text{tangent bundle of } U_1) \oplus \xi \cong \varepsilon^n \oplus \xi.$$

Thus M_1' is not parallelizable. This completes the proof of Corollary 1.

COROLLARY 2. *There exists a topological manifold M_2 such that no Cartesian product $M_2 \times M'$ can be given a smoothness structure.*

Sketch of proof. Let U_2 be an open subset of some euclidean space R^n having the homotopy type of X_2. Then there exists a microbundle $\mathfrak{x}$ over U_2 whose s-class does not belong to the image of the homomorphism

$$k_{\mathrm{Orthog}} U_2 \to k_{\mathrm{Top}} U_2.$$

Let M_2 be the total space of this microbundle. We may assume that M_2 is a manifold.

It can be shown that the tangent microbundle of M_2, restricted to U_2, is isomorphic to the Whitney sum

$$(\text{tangent microbundle of } U_2) \oplus \mathfrak{x} \cong \mathfrak{e}^n \oplus \mathfrak{x}.$$

Since this is not isomorphic to $|\xi|$ for any vector bundle ξ over U_2, it follows from the Corollary to Theorem 1 that M_2 is not smoothable.

Given any positive integer p, a similar argument shows that the product $M_2 \times R^p$ is not smoothable. But this implies that no Cartesian product $M_2 \times M'$ can be smoothable; and proves Corollary 2.

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