BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 52, Number 3, July 2015, Pages 539–542
S 0273-0979(2015)01490-4
Article electronically published on March 10, 2015

Geometry of isotropic convex bodies, by Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou, Mathematical Surveys and Monographs, Vol. 196, American Mathematical Society, Providence, RI, 2014, xx+594 pp., ISBN 978-1-4704-1456-6

A priori, one would expect geometry in high-dimensional spaces to be rather complicated. Our experience in two and three dimensions seems to indicate that as the number of dimensions increases, the number of possible configurations grows rapidly, and we enter the realm of enormous, unimaginable diversity. For instance, the classification of closed two-dimensional surfaces has been known for a long time, and around a dozen years ago Perelman completed the proof of Thurston's geometrization conjecture regarding the structure of closed three-dimensional manifolds. However, it is known that the space of all closed manifolds in dimension $n \geq 4$ is quite complex, with little or no hope for structural theorems of the type available in lower dimensions. The immediate conclusion appears to be the "curse of dimensionality", that a general, interesting theory is unlikely to exist in high-dimensional geometry.

The book under review promotes the contrary opinion: that dimensionality, when correctly viewed, may become a blessing. One of the earliest examples of simplicity and universality in high dimensions is the classical central limit theorem: The normalized sum of independent random variables is approximately Gaussian, under quite general assumptions, when the number of variables approaches infinity. In other words, for large n,

$$\frac{e^{-t^2/2}}{\sqrt{2\pi}} \approx \int_{H_t} \prod_{i=1}^{n} f_i(x_i) d\lambda_{n-1}(x)$$

for $H_t = \{x \in \mathbb{R}^n \,;\, \sum_{i=1}^n x_i = t\sqrt{n}\}$, whenever $f_1, \ldots, f_n$ are probability densities on the real line of mean zero and variance one, satisfying certain mild regularity conditions. Thus, the specific form of the Gaussian density emerges in the limit $n \to \infty$ when integrating rather arbitrary product densities in $\mathbb{R}^n$ on hyperplanes. Another example for harmony and order in high dimensions is Dvoretzky's theorem, which asserts that any high-dimensional convex body has nearly Euclidean sections of a large dimension. Note that the symmetries of the Euclidean ball appear, even though we made only minimal assumptions: just convexity and the high dimension. The central limit theorem and Dvoretzky's theorem are high-dimensional effects that lack clear analogs in low dimensions. A third example is Ramsey's theorem in graph theory, which demonstrates that any coloring of a large complete graph by a fixed number of colors contains a sizable, monochromatic, induced subgraph.

As it turns out, there are motifs in high-dimensional geometry which seem to compensate for the difficulties that arise from high dimensionality. One of these motifs is the *concentration of measure phenomenon*. Quite unexpectedly, a scalar Lipschitz function on a high-dimensional space behaves in many cases as if it were a constant function. For example, if we sample five random points from the

2010 *Mathematics Subject Classification.* Primary 52Axx, 46Bxx, 60Dxx, 28Axx.

n-dimensional unit sphere, for large n, and substitute them into a 1-Lipschitz function, then we will almost certainly obtain five numbers that are very close to one another. This phenomenon is reminiscent of the well-known geometric property that in the high-dimensional Euclidean sphere, "most of the mass is close to the equator, for any equator". This geometric property, which follows from the isoperimetric inequality, is unthinkable in, say, three dimensions. Since Milman's proof of Dvoretkzy's theorem in the 1970s, the concentration of measure has become a major tool in high-dimensional analysis.

The book by Brazitikos, Giannopoulos, Valettas, and Vritsiou is dedicated to the study of volume distribution in high dimensions. Its central idea, as I understand it, is that the spatial arrangement of volume due to the geometry of $\mathbb{R}^n$, for large n, imposes rigidity on convex sets and convexity-related measures. Convexity is one of the ways in which one may harness the concentration of measure phenomenon in order to formulate clean, nontrivial theorems. The Brunn–Minkowski inequality from the end of the 19th century states that for any nonempty Borel sets $A, B \subseteq \mathbb{R}^n$,

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{1}$$

where $A + B = \{x + y\,; x \in A, y \in B\}$ and $|A|$ is the volume of the set A. The Brunn–Minkowski inequality is a close relative of the isoperimetric inequality, and equality holds in (1) essentially only when A and B are congruent convex bodies. Among the many consequences of the Brunn–Minkowski theory, let us mention the reverse Hölder inequalities, proven by Berwald in the 1940s and Borell in the 1970s, stating that for any convex body $K \subseteq \mathbb{R}^n$, a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ and $p, q > 0$,

$$\left(\int_K |f(x)|^p \frac{dx}{|K|} \right)^{1/p} \leq C \left(\int_K |f(x)|^q \frac{dx}{|K|} \right)^{1/q}, \tag{2}$$

where $C = C_{p,q} > 0$ depends solely on p, q and neither on K nor on the dimension n. This amusing property of convex domains goes beyond linear functionals, as it remains valid when f is merely a positive, concave function, and also when f is an arbitrary polynomial of degree d in n real variables. In the latter case, the constant C depends only on p, q and d, as shown by Bourgain in the early 1990s. These results serve as evidence for the general hypothesis, that in many respects the uniform measure on a high-dimensional convex body resembles a Gaussian measure. Recall that Gaussian random variables are completely determined by their mean and variance. The barycenter and the covariance matrix of a convex body $K \subseteq \mathbb{R}^n$ are defined by

$$b(K) = \int_K x \frac{dx}{|K|}, \qquad \mathrm{Cov}(K) = \int_K (x - b(K)) \otimes (x - b(K)) \frac{dx}{|K|},$$

where $x \otimes x = xx^t = (x_i x_j)_{i,j=1,\ldots,n}$ for $x \in \mathbb{R}^n$. A convex body is said to be isotropic if its barycenter is at the origin and its covariance matrix is scalar. Any convex body has an affine image which is isotropic. If our goal is to test the aforementioned hypothesis, then it suffices to restrict our attention to isotropic convex bodies, as already indicated by the title of the book. Let us provide another regularity result from the book, published by Hensley in 1980: When $K \subseteq \mathbb{R}^n$ is convex and isotropic, for any two linear subspaces $H_1, H_2 \subseteq \mathbb{R}^n$ with $\dim(H_1) = \dim(H_2) = n - 1$,

$$|K \cap H_1|_{n-1} \leq C |K \cap H_2|_{n-1},$$

where $|\cdot|_{n-1}$ stands for $(n-1)$-dimensional volume, and $C > 0$ is a universal constant which is at most $\sqrt{6}$ according to Ball and to Fradelizi. However, the flow of positive results soon approaches an obstacle, since the following seemingly innocent question is yet undecided: Given a convex set $K \subseteq \mathbb{R}^n$ of volume one, does there exist a hyperplane $H \subseteq \mathbb{R}^n$ such that the $(n-1)$-dimensional volume of $K \cap H$ is at least c, for some universal constant $c > 0$? This question, widely known as the slicing problem or the hyperplane conjecture, was raised by Bourgain in the 1980s. It may be reformulated in several equivalent ways, and it lurks in the background of many of the chapters of the book.

The study of the uniform measure on high-dimensional convex bodies has been particularly fruitful in the past decade. This new era has begun with the Paouris inequality from 2005, establishing a large deviation estimate for the Euclidean norm on isotropic convex bodies, which was much stronger and more general than any other estimate known prior to this result. Soon afterwards, weak forms of the slicing problem were established, as well as the existence of nearly sub-Gaussian tail estimates for the distribution of linear functionals on arbitrary convex sets. Next in line came the central limit problem for convex sets, proposed by Anttila, Ball, and Perissinaki and by Brehm and Voigt in the late 1990s. This problem is now largely resolved, thanks to the following central limit theorem for convex sets from 2006: Let X be a random vector in $\mathbb{R}^n$ that is distributed uniformly in an isotropic convex body. Assume the normalization $\mathbb{E}|X|^2 = n$. Then there exists a unit vector $\theta \in \mathbb{R}^n$ such that

$$\left| \mathbb{P}\left[\langle X, \theta\rangle \leq t\right] - \int_{-\infty}^{t} \frac{\exp(-s^2/2)}{\sqrt{2\pi}} ds \right| \leq \frac{C}{n^\alpha} \qquad \text{for all } t \in \mathbb{R}, \tag{3}$$

where $C, \alpha > 0$ are universal constants, and $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^n$. In fact, most unit vectors θ in the unit sphere S^{n-1} satisfy (3), where by "most" we refer to the uniform probability measure on S^{n-1}. Essentially, the proof of the central limit theorem for convex sets reduces to the following *thin-shell* bound:

$$\mathrm{Var}\left(\frac{|X|}{\sqrt{n}}\right) \leq \frac{\tilde{C}}{n^\beta}, \tag{4}$$

where $\tilde{C}$ and $\beta \approx 2\alpha$ are positive universal constants, $|\cdot|$ is the standard Euclidean norm, and Var stands for the variance of a random variable. The bound (4) implies that 99% of the mass of an isotropic convex body in high dimensions is located in a thin spherical shell, whose width is much smaller than its radius. If the convex body is not necessarily isotropic, then most of its measure is very close to a surface of an ellipsoid. The optimal exponents α and β from (3) and (4) are still unknown. The current world record is $\beta \geq 1/3$, held by Guédon and E. Milman as of 2011, succeeding Fleury who, in 2009, obtained the bound $\beta \geq 1/4$. The thin-shell conjecture speculates that $\beta = 1$. The thin-shell conjecture implies the hyperplane conjecture, according to Eldan and the reviewer. Furthermore, if the thin-shell exponent β equals one, then we would have a satisfactory rate of convergence in the central limit theorem for convex bodies, which is comparable to the Berry–Esseen theorem from probability theory. Therefore, the question about the best value of β tests whether convexity assumptions are as good as independence assumptions in the context of the central limit theorem. The question of estimating the thin-shell exponent is also closely related to spectral gap of the Laplacian and to isoperimetric inequalities in convex sets. Suppose that $K \subseteq \mathbb{R}^n$ is a bounded, open convex set of volume one. Its isoperimetric constant is defined as

$$h_K = \inf \left\{ |\partial A \cap K|_{n-1} \, ; \, |A \cap K|_n = \frac{1}{2} \right\}, \tag{5}$$

where the infimum runs over all domains $A \subseteq \mathbb{R}^n$ with smooth boundary, that capture half of the mass of K. Thus, h_K is the area of the minimal interface which is required in order to divide K into two parts of equal mass. The Kannan–Lovász–Simonovits (KLS) conjecture from the 1990s suggests that the infimum in (5) is equivalent, up to a universal constant, to the same infimum where A ranges only over the collection of half-spaces in $\mathbb{R}^n$. The KLS conjecture would immediately imply the thin-shell conjecture, and in 2012 Eldan showed that these two conjectures are intimately related and that, up to logarithmic factors, they are equivalent.

The book spans a lot more than the specific results and conjectures that were selected to be presented above. It contains a background chapter on asymptotic convex geometry, which will be expanded in the forthcoming book by Artstein-Avidan, Giannopoulos, and Milman on asymptotic geometric analysis. The Paouris theory of the L^p-centroid bodies is systematically developed, as well as the theory of the logarithmic Laplace transform in convexity and the Bourgain–Milman inequality. Various reductions, partial answers, and equivalent formulations of the above conjectures are discussed. The treatment is comprehensive, including analysis of relations to information theory, random polytopes and infimum convolution inequalities. The authors provide a thorough account of the current state of the field, and their arguments are detailed and accessible to beginning graduate students. Prior to the publication of this book, results in the field were scattered in many research articles, and the book does a remarkable job in unifying and presenting old contributions alongside the new.

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