BOOK REVIEWS

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Mathematics & climate, by Hans Kaper and Hans Engler, SIAM, Philadelphia, PA, 2013, xx+295 pp., ISBN 978-1-611972-60-3

Applied mathematicians and fluid dynamicists have worked together for centuries since the equations of motion were formulated by Euler. In the first decades of the twentieth century it was recognized by mathematician and meteorologist L. F. Richardson that the atmospheric motions (weather) were governed by a set of equations based upon Newton's laws of mechanics that could in principle be solved by numerical integration, and Richardson attempted to carry out a forecast by hand calculations. It was soon recognized that high-frequency components (e.g., sound waves and some gravity-driven waves) had to be filtered out to satisfactorily bring out the variance in the time scales of weather-related motions. In the late 1940s theoretical meteorologist J. Charney and mathematician J. von Neumann worked together in the numerical implementation of such a prediction project on the very first digital computer constructed at the Institute for Advanced Study in Princeton, NJ. Steady progress, primarily in increasing space-time resolution, in weather-forecasting skill has been intimately tied to improvements in computer technology (both speed and capacity) and numerical methods.

The roots of modern climate science extend back to the late 1960s as numerical weather forecasting was reaching maturity. One could run the old initial-value-driven forecast model for years and generate a model climate, probably first undertaken by theoretical meteorologist N. Phillips. This required augmentation of the weather model by including radiation physics and other subtle features that might change gradually over time scales of months to decades to centuries. This was happening as "climatology" was evolving into "climate science". This evolution constituted a paradigm shift as climate was recognized to be a physics problem that could be solved through the use of fluid dynamics, radiative transfer physics, meteorology, oceanography, applied mathematics, statistics, and numerical methods. As in all paradigm shifts many "old fashioned climatologists" were reluctant to buy in. One of the first lessons of the new paradigm was that climate (i.e., all the statistical properties, including means, variances, and covariances in space and time) of weather in the atmosphere and the ocean could actually change given a change in the balance of external radiation fluxes, such as a change in the sun's brightness.

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Today's climate models grind out century-long simulations at very high spatial resolution (of the order of $100 \times 100\,\mathrm{km}^2$ grid boxes). One problem is that the data fields are so large and complex that they are nearly as hard to interpret as the real observational data. From the beginning of climate science there emerged a family of simpler models that were based on the most elementary principles, such as the imbalance of radiation fluxes at the top of the atmosphere. They included simple phenomenological global models based on fitting radiation data to simplified formulae (called *parametrization* by the players). Two such implementations, including latitude dependence, were devised by the Soviet scientist M. I. Budyko and the American W. D. Sellers. These latter models used simplified forms for cross-latitude heat fluxes due to transport by the atmosphere and oceans. The importance of the planetary reflectivity (*albedo*) was found to be key to the radiation balance. When an idealized ice cap was introduced whose equator-ward extent was tied to the latitude of a freezing temperature (nominally -10°C), the size of the ice cap would be dependent on the temperature field. The reflected solar radiation then depended on ice-cap size. The net result was a strong nonlinear positive feedback mechanism. One disturbing finding was that if the sun's brightness was decreased by only a few percent, the Earth's surface temperature would plunge, and Earth would become an ice-covered planet, referred to as a "Snowball Earth". The transition to Snowball Earth was shown to be a bifurcation in the operating curve (ice-cap size versus solar brightness). The tractability of the simpler models attracted the attention of mathematicians, such as Michael Ghil and others, with mathematical physics skills (e.g., I. Held, M. Suarez, and also this reviewer) who studied the bifurcation properties and the stability of the solution branches. The role of simplified models in climate science is to aid in revealing some of the more dominant aspects of the problem, which might be obscured by the sheer volume of output produced by simulations of more complex models.

Solutions to the "energy balance models" are for the ensemble average of climatological fields. Individual realizations of the climate system rattle along in a stochastic fashion about the ensemble mean. One can generate an ensemble of solutions by running the individual ensemble members with slightly different initial conditions. This brings us to another branch of mathematics wherein there was a store of expertise, for instance, Russian mathematicians such as Kolmogorov and his academic progeny, Obukhov, Monin, Yaglom, and Golytsin, who had made earlier spectacular contributions in turbulent flows and random fields. In my opinion, the climate system, especially the surface temperature field, acts for many purposes like a Gaussian random field. Many of the statistical features of this field are governed by rather simple autoregressive-like linear models. The noise-forcing term (weather) is nearly white noise compared to the larger-scale slow response of the temperature field. The result is red noise, which was first applied to the climate problem by K. Hasselmann.

The random field interpretation of climate fields is useful in many aspects of data representation and interpretation. Decomposing a random field into a set of orthogonal basis functions (the so-called Empirical Orthogonal Functions, or EOFs) whose random coefficients are statistically independent has become a very important means of analyzing data. The EOFs are generally patterns on a map of the whole Earth or of regions of it. The time series of coefficients of the EOF patterns are known in other disciplines as the principal components. The EOFs are actually sample estimates of the Karhunin–Loève functions of random field theory.

Another area in climate science that has interested mathematicians over the years is physical oceanography, which plays a key role in the climate system because of its contribution to variability at long time scales longer than that of weather. There are many simplified models in oceanography, especially those devised by H. Stommel and W. Monk. These have led to a number of solvable (or nearly so) models exhibiting solutions with curious bifurcation structures.

Perhaps the largest problem where mathematical statistics has influenced climate science is the recognition of the inadequacy of sample size in most data sets. We only have so much data, and presumably Earth's climate records only constitute one realization of it. Moreover, we like to treat it as a stationary (and ergodic) time series, but especially when disturbed from the outside it might not be so. In practice we never have enough data, and this leads to many mistaken conclusions by those who are not careful in the number of independent samples that are included in a particular problem or test. One of these is regressions of one data set on another. A stream of data from daily or yearly or decadally averaged values are often serially correlated, and this means the number of pieces of independent information is often much fewer than the careless analyst thinks. Such an error will lead to unrealistically optimistic outcomes in formal statistical tests.

The book *Mathematics & climate* by Hans Kaper and Hans Engler is a successful attempt in a brief survey to bring the world of climate science to the attention of aspiring and mature mathematicians, physicists, and engineers. The book covers many mathematical methods that are used in climate science, such as advanced differential equations, eigenvalue analysis, time delay differential equations, Fourier analysis, bifurcations, stability, and latitudinal special functions on the sphere. Numerical analysis is passed over—I suppose because it is expected to have been learned elsewhere in the curriculum.

No book covering climate science could succeed without covering statistical methods, including correlation, autocorrelation, regression, extreme value statistics, and data analysis. This book nicely broaches these subjects. The authors even explain the technique of data assimilation, wherein an observed data set (including errors and gaps in space-time) is massaged (or filtered) in such a way as to optimize forecasts. Such techniques are also useful in reconstructing atmospheric fields from many years past. The current book only briefly mentions stochastic models *per se.*

The book succeeds in describing a number of interesting models (sometimes called "Toy Models") of the ocean circulation, biogeochemical processes, and ice-cap-albedo energy balance models. Many different geophysical subjects are touched upon briefly, but accurately. These include models of the El Niño–Southern Oscillation (ENSO). These beautiful formulations would whet the appetite of anyone starting to learn the subject of climate science. Other topics of interest to mathematicians and statisticians are elegantly reviewed, including the Lorenz model and its role in limiting predictability. I have found no errors in any of the topics discussed involving physics, chemistry, or climate science. I commend the authors for getting to know and understand all of this diverse material and packaging it for the intended audience.

Some chapters in the book are descriptive and have almost no mathematics. These are necessary to convey the spirit of climate science which has expanded to include just about everything in physics, chemistry, and now biology. The sheer range of topics naturally leads to some unevenness of the presentation. Every chapter has an abundance of exercises for the students. I am sure much can be learned in working these out, and they surely lead to interesting discussions.

I recommend the book not only as a text for mathematics students at the advanced undergraduate or beginning graduate level, but also to those well into their careers. I believe that many engineers, chemists, and others with good mathematics backgrounds would also enjoy reading the book. Many graduate programs in meteorology or oceanography would welcome a good new graduate student versed in these topics into their climate programs.

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