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*Topics in occupation times and Gaussian free fields*, by Alain-Sol Sznitman, EMS Textbooks in Mathematics, European Mathematical Society, Zürich, 2012, viii+114 pp., ISBN 978-3-03719-109-5, €28.00

A continuous time Markov process on a finite or countable state space $\mathcal{X}$ that is reversible with respect to the counting measure has transition probabilities $p(t,x,y)$ that satisfy $p(t,x,y) = p(t,y,x)$. With a positive killing rate if the process is recurrent, the Green's function $g(x,y) = \int_0^\infty p(t,x,y)dt$ is well defined and is a symmetric positive definite function. There is a natural Gaussian measure $P^G$, the distribution of the collection $\{\xi_z\}$ indexed by $z \in \mathcal{X}$ with $E[\xi_z] = 0$ and $E[\xi_x\xi_y] = g(x,y)$. Its probability density (with respect to Lebesgue measure on $\mathbb{R}^{\mathcal{X}}$) can be written as

$$c \exp\big[\frac{1}{2}\sum_{x,y\in\mathcal{X}} k(x,y)\xi_x\xi_y\big].$$

While this makes sense if $\mathcal{X}$ is finite, this is just a formal expression if $\mathcal{X}$ is infinite. The matrix $k(x,y)$ is the infinitesimal generator

$$k(x,y) = \frac{d}{dt}p(t,x,y)|_{t=0} = \lim_{t\to 0}\frac{p(t,x,y)-\delta_{x,y}}{t},$$

and it is negative definite. The matrices $g$ and $-k$ are inverses of one another. It is also clear that $k(x,y) \geq 0$ for $x \neq y$ and $k(x,x) + \sum_y k(x,y) = -V(x) \leq 0$ and $V(x)$ would be the killing rate at $x$.

The picture is that of a Markov chain that waits for an exponential time and jumps at rate $-k(x,x)$, i.e., the probability that the waiting time at $x$ is greater than $T$ is $e^{k(x,x)T}$. Then it jumps to a randomly chosen state $y$ with probability $\frac{k(x,y)}{-k(x,x)}$ and starts afresh. If $V(x) > 0$, then $1 - \sum_y \frac{k(x,y)}{-k(x,x)} = \frac{V(x}{-k(x,x)}$ is the probability of being killed or jumping to a *cemetery* state $\Delta$ which it never leaves. The process jumps there when killed and stays there for ever and $k(x,\Delta) = V(x)$. In the transient case it naturally wanders off to $\infty$ and the killing is not needed, i.e., $V(x)$ can be 0. $P_x$ denotes the probability measure on the trajectories corresponding to the chain.

There are deep connections between the Gaussian field and the Markov chain. These involve the local times of the Markov chain, which are defined as

$$\ell_z^\zeta = \int_0^\zeta \mathbf{1}_z(x(s))ds$$

or the time spent at the site $z$ by the process $x(s)$ until time $\zeta$ which is the killing time if the process is killed or $\infty$ if it is transient and wanders off naturally. In any case $\mathbf{1}_z(x(s)) = 0$ for $s > \zeta$ for $z \in \mathcal{X}, z \neq \Delta$.

In the probabilistic context we have Eisenbaum and Dynkin isomorphisms that are known as Ray–Knight type theorems. They involve the measures $P_x$ or modifications $P_{x,y}$ defined by

$$P_{x,y}[x(t_1) = x_1, \ldots, x(t_n) = x_n] = \int_{t_n}^\infty P_x[x(t_1) = x_1, \ldots, x(t_n) = x_n, x(t) = y]dt$$

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with total mass $g(x,y)$ that live on the space of paths of some finite random duration, that begin at $x$ and get killed at $y$.

The Dynkin isomorphism theorem [2] says that for any $x,y \in \mathcal{X}$ the joint distribution of the collection $\{\ell_z^\zeta + \frac{1}{2}\xi_z^2\}_{z\in\mathcal{X}}$ under $P_{x,y} \otimes P^G$ is the same as that of $\{\frac{1}{2}\xi_z^2\}$ under $\xi_x\xi_y P^G$, which is a signed measure if $x \neq y$. Notice that the total mass of either one is $g(x,y)$.

Eisenbaum's isomorphism [3] uses only $P_x$ rather than $P_{x,y}$. According to it, for $s \neq 0$, the joint distribution of $\{\ell_z^\zeta + \frac{1}{2}(\xi_z+s)^2\}_{z\in\mathcal{X}}$ under $P_x \otimes P^G$ is the same as that of $\{\frac{1}{2}(\xi_z+s)^2\}_{x\in\mathcal{X}}$ under the signed measure $(1+\frac{\xi_x}{s})P^G$.

Then there is the Symanzik representation [6] of even order moments $\prod_{i=1}^{2n} \xi_{z_i}$ of the Gaussian field modified by

$$c\Big[\prod_{z\in\mathcal{X}} g\big(\frac{\xi_x^2}{2}\big)\Big] dP^G$$

with $c$ being the normalizing constant. The representation involves "a gas of loops". A path $x(s)$ with $x(t) = x(0)$ defines a rooted loop $\widetilde{\omega}$ of size $t$ with $x(0) = x(t)$ being the root. The local time $\ell_z(\widetilde{\omega})$ is the integral $\int_0^T \mathbf{1}_z(x(s))ds$, i.e., the time spent by the loop at $z$. Let $\Omega_t$ be the space of all rooted loops of size $t$, and let $\Omega = \bigcup_{t>0}\Omega_t$. A Poisson point process $Q$ on the space $\Omega$ is determined as $\int_0^\infty \lambda_t(B)dt$, where $\lambda_t(\cdot)$ is an intensity measure on $\Omega_t$ constructed in a natural way from the underlying Markov chain. The representation of Symanzik involves the ratio of

$$\sum_{\mathcal{P}} E^{P_{x_1,y_1}} \cdots E^{P_{x_n,y_n}} E^Q\Big[\prod_{z\in\mathcal{X}} \mu\Big(\sum_{\widetilde{\omega}} \ell_z(\widetilde{\omega}) + \sum_i \ell_z(\omega_i)\Big)\Big]$$

to

$$E^Q\Big[\prod_{z\in\mathcal{X}} \mu\Big(\sum_{\widetilde{\omega}} \ell_z(\widetilde{\omega})\Big)\Big],$$

where $\mu(\cdot)$ involves the moments of $g$. The set $(z_1,\ldots,z_{2n})$ is partitioned into $n$ pairs $(x_i,y_i)$ and the summation over $\mathcal{P}$ is over all possible partitions. $\omega_i$ is the variable of integration for $P_{x_i,y_i}$, and $\{\widetilde{\omega}\}$ is the random collection of loops whose statistics is given by $Q$. Symanzik's idea was further developed by Brydges, Frolich, and Spencer in [1].

In these notes, based on lectures given in ETH, Zürich, the author develops the theory of Markov chains reversible with respect to a weight, covering several aspects including some potential theory. Standard material such as local times and the Feynman–Kac formula are dealt with. The connection between the Green's function and the associated Gaussian random field is explored in detail. The isomorphism theorems of Eisenbaum and Dynkin are presented in detail and connections to Ray–Knight theorems [5], [4] are pointed out.

In the next section suitable intensity measures for a Poisson gas of loops are introduced and the Symanzik representation is established. Finally in the last section, in the transient case, a limit is taken where the root of the loop is pushed to $\infty$ and the intensity increased and fine tuned, so that the probability of at least one loop passing through a given point remains nontrivial. In the limit this produces a random family of loops "passing through" $\infty$ and their properties are studied. For example, the probability that the gas of loops avoids a given set is calculated.

The notes are quite detailed and self-contained.

## References


[1] David Brydges, Jürg Fröhlich, and Thomas Spencer, *The random walk representation of classical spin systems and correlation inequalities*, Comm. Math. Phys. **83** (1982), no. 1, 123–150. MR648362 (83i:82032)

[2] E. B. Dynkin, *Markov processes as a tool in field theory*, J. Funct. Anal. **50** (1983), no. 2, 167–187, DOI 10.1016/0022-1236(83)90066-6. MR693227 (85e:81074)

[3] Nathalie Eisenbaum, *Dynkin's isomorphism theorem and the Ray–Knight theorems*, Probab. Theory Related Fields **99** (1994), no. 2, 321–335, DOI 10.1007/BF01199028. MR1278888 (95d:60128)

[4] F. B. Knight, *Random walks and a sojourn density process of Brownian motion*, Trans. Amer. Math. Soc. **109** (1963), 56–86. MR0154337 (27 #4286)

[5] Daniel Ray, *Sojourn times of diffusion processes*, Illinois J. Math. **7** (1963), 615–630. MR0156383 (27 #6306)

[6] Symanzik, Kurt, *Euclidean Quantum Field Theory.* In Scuola Internazionale di Fisica, "Enrico Fermi" XLV Corso, Academic Press, 1969, 152–223.


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