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ABOUT THE COVER: SOPHIE GERMAIN AND A PROBLEM IN NUMBER THEORY

GERALD L. ALEXANDERSON

Marie-Sophie Germain (1776–1831), a largely self-educated French mathematician, for many years has been known for her early work in number theory and also for her work in physics, where she studied vibrating plates and Chladni figures as well as related questions in the theory of elasticity. Though much of her work in number theory was on Fermat's Last Theorem, she, of course, failed to prove it. But recently examined manuscripts and letters have revealed that between 1816 and 1819 she went much more deeply into that problem than had previously been known. In 2010, R. Laubenbacher and D. Pengelley described this discovery in an important paper in *Historia Mathematica* [5]. The authors claim that there is now evidence, if any more were needed, that Germain was the most productive contributor to mathematics among women up to her time, where others might have been Hypatia, Maria Gaetana Agnesi, or Émilie du Châtelet.

As important as her mathematical work is, however, the more inspiring story about Germain is that she carried out this work after a childhood in Paris in the late eighteenth century where, as a young woman, she was strongly discouraged from studying mathematics and was not allowed to attend the École Polytechnique. With enormous determination, she persisted in studying mathematics when she was in her teens by reading mathematics books in her father's library. We are told in G. Libri's early biography of Germain [6] that even her parents tried to discourage her from studying mathematics; they did not think that it was a suitable area of study for a young lady in late eighteenth century French society. Libri provides stories of the young Germain working on mathematics well into the night even after there was no heat from the fireplaces and her room was so cold that the ink froze in the inkwells. The story of Germain's persistence in studying mathematics is told by D. Musielak in a recent work, *Sophie's Diary/A Mathematical Novel* [7]. The story is a fascinating one. Germain, a member of a bourgeois family of silk merchants,

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FIGURE 1. Sophie Germain

spent her early years in the turbulent Paris of the 1790s. Since her father held various governmental positions and witnessed some of the worst excesses of the French Revolution—the storming of the Bastille and the subsequent executions of Louis XVI and Marie-Antoinette—his family could easily have become victims of the new revolutionary regime. But they survived.

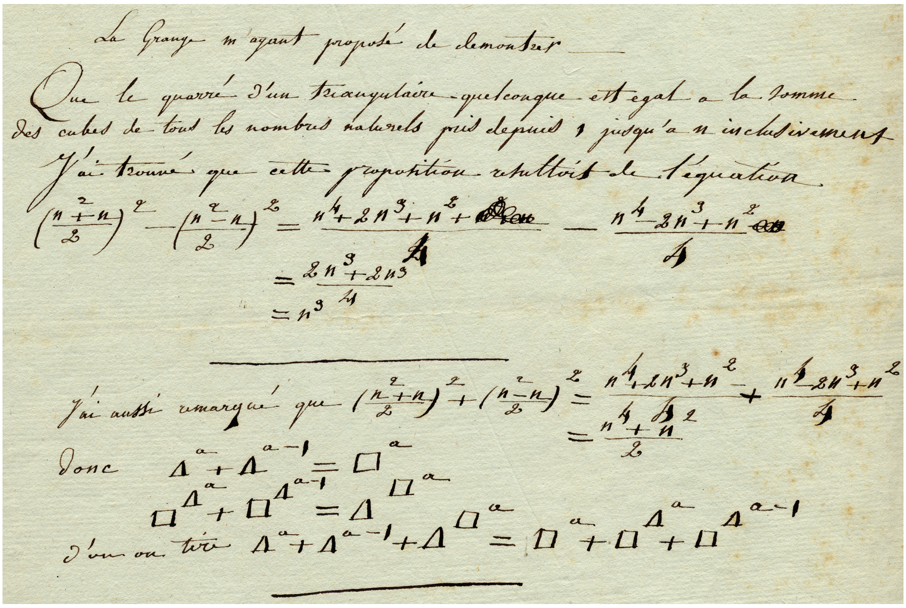
Germain continued to work on number theory until 1819, well after those cataclysmic events. She wrote regularly to Legendre about her efforts to develop a grand plan to prove Fermat's Last Theorem (FLT), proving along the way an important special case today called Germain's Theorem. Legendre was at the same time working on his own attempt at a proof but was sufficiently impressed by Germain's work that he alluded to it in an 1825 supplement to the second edition of his *Essai sur la Théorie des Nombres* (1808). During that period she carried on extensive correspondence with Gauss after the appearance of his *Disquisitiones Arithmeticae* in 1801. Of course we now know that FLT was not going to be proved by methods then available and a solution had to await the development of far more sophisticated techniques, with which it was finally proved by Andrew J. Wiles and Richard Taylor in 1995.

How is it that so little was known of Germain's contributions until very recently, with the appearance of the work of Laubenbacher and Pengelley and some simultaneous investigations by Andrea Del Centina [1]? This question brings us back to Germain's friend, the fascinating Count Guglielmo Libri-Carucci (1803–1869), who wrote her obituary [6] in 1832, a few months after her death. As a competent mathematician and as someone also interested in FLT, he later wrote that Germain's approach could not have led to a solution of the problem. But the story is more complicated. In [5] we read in a footnote that Libri was a "mathematician, historian, bibliophile, thief and friend of Sophie Germain." That's a generous appraisal of the man's career. (Of course, seeing that he was a bibliophile does remind us that the count was aptly named—"Count Books"!) Though he was a mathematician sufficiently competent to have been appointed in 1823 to the chair of mathematical physics at the University of Pisa, today his reputation, among bibliophiles, at least, rests mainly on his having been a thief. As recently as February 2010 the *New York Times* reported the return of a letter of Descartes dated May 27, 1641, from the Haverford College Library in Pennsylvania to France, a letter Libri had stolen from the Institut de France in the 1840s [2]. (The *Times* had some fun with the title of its report: "Descartes Letter Found. Therefore It Is.") For political reasons this Italian count from Tuscany had fled to Paris in the early 1830s, where he continued to correspond with some of the most active mathematicians and scientists of the day. He wrote on the history of mathematics, working his way up in French intellectual circles to election to the Académie des Sciences, Paris, in 1833, where he succeeded Legendre. But, given access to some of the greatest French collections of books and manuscripts on the history of mathematics, he was eventually accused of stealing material to build up his own personal collection, which amounted to some 30,000 books. When a warrant was issued for his arrest he fled to London in 1848. There, having shipped 17 large cases of books to England, he proceeded to sell much of the collection in collusion with some London dealers, and it was only when faced with charges there that he departed for Italy in 1868 with much of his collection of letters and manuscripts still intact. Lest we give him too much of the benefit of the doubt—that he was just such a passionate bibliophile he could not resist the temptation of adding to his collection by any means available—we should note that there is evidence that his efforts were often decidedly criminal, and he "arranged with binders, forgers, and others to change the copies by adding false ownership signatures . . . altering and obliterating library markings, fabricating receipts and purchase records, &c." [2].

There is, however, a case to be made for Count Libri. Since Germain did not hold any academic appointment or membership in a scientific academy, there was no obvious recipient of her collected papers upon her death, but through Libri they ended up in the collections of the Bibliothèque Nationale in Paris and a smaller number in the Biblioteca Moreniana in Florence, quite likely those that he had claimed for himself. It is largely the survival of these manuscripts that has made possible some of the recent scholarship on Germain and the discoveries of her work on FLT.

The modest undated manuscript shown on the cover of the *Bulletin of the American Mathematical Society* this month (see Figure 2) records an observation by Germain on what is referred to as a problem posed by Lagrange. The dealer who sold the manuscript in the 1990s was the highly respected London firm, Bernard Quaritch, which dated it as circa 1797. On occasion Germain when corresponding with mathematicians signed the letters "M. LeBlanc," assuming, probably correctly, that had it been known that these letters on mathematics were from a woman they would have been ignored. Perhaps the pseudonym was chosen because there had been a student at the École Polytechnique, Antoine Auguste LeBlanc, a year older than Germain, who had died young [4]. There is no signature on this manuscript.

In it Germain shows that for a certain sum of three triangular numbers from the set $\{1, 3, 6, 10, 15, 21, \dots\}$, that sum can be expressed as the sum of three squares. Problems of this kind had been around for many years. L. E. Dickson pointed out that Pythagoras was aware of triangular numbers [3]. Sums of squares had been studied by the Chinese in the thirteenth century. In 1636 Fermat stated a general theorem on sums of triangular numbers, squares, and pentagonal numbers. This was attributed to St. Croix and was much discussed by Descartes and Mersenne. But it was not until 1654 that Pascal noted that the sum of triangular numbers could be read from his eponymous triangle. Well after Germain's death, people were still discovering theorems of this type, for example, that every integer is the sum of a square and two triangular numbers, or the sum of two squares and a single triangular number, attributed by Dickson to E. Lionnet, V. A. Lebesgue, and S. Réalis (1872). So Germain's interest in this very modest result is not surprising.


La Grange m'ayant proposé de demontrer —

Que le quarré d'un triangulaire quelconque est egal a la somme des cubes de tous les nombres naturels pris depuis 1 jusqu'a n inclusivement

J'ai trouvé que cette proposition resultoit de l'equation

$$\left(\frac{n^2+n}{2}\right)^2 - \left(\frac{n^2-n}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4} - \frac{n^4-2n^3+n^2}{4}$$
$$= \frac{2n^3+2n^3}{4}$$
$$= n^3$$

J'ai aussi remarqué que $\left(\frac{n^2+n}{2}\right)^2 + \left(\frac{n^2-n}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4} + \frac{n^4-2n^3+n^2}{4}$

$$= \frac{n^4+n^2}{2}$$

donc $\triangle^a + \triangle^{a-1} = \Box^a$

$\Box^{\triangle^a} + \Box^{\triangle^{a-1}} = \triangle^{\Box^a}$

d'où on tire $\triangle^a + \triangle^{a-1} + \triangle^{\Box^a} = \Box^a + \Box^{\triangle^a} + \Box^{\triangle^{a-1}}$

FIGURE 2

Here is the text:

La Grange m'ayant proposé de démontrer—

Que le quarré d'un triangulaire quelconque est égal à la somme des cubes de tous les nombres naturels pris depuis 1 jusqu'à n inclusivement.

J'ai trouvé que cette proposition résultais de l'équation

$$\left(\frac{n^2+n}{2}\right)^2 - \left(\frac{n^2-n}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4} - \frac{n^4-2n^3+n^2}{4}$$
$$= \frac{2n^3+2n^3}{4}$$
$$= n^3.$$

J'ai aussi remarqué que

$\left(\frac{n^2+n}{2}\right)^2 + \left(\frac{n^2-n}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4} + \frac{n^4-2n^3+n^2}{4} = \frac{n^4+n^2}{2}$

donc $\triangle^{\boldsymbol{a}} + \triangle^{\boldsymbol{a-1}} = \Box^{\boldsymbol{a}}$

$\Box^{\triangle^{\boldsymbol{a}}} + \Box^{\triangle^{\boldsymbol{a-1}}} = \triangle^{\Box^{\boldsymbol{a}}}$

d'où on tire $\triangle^{\boldsymbol{a}} + \triangle^{\boldsymbol{a-1}} + \triangle^{\Box^{\boldsymbol{a}}} = \Box^{\boldsymbol{a}} + \Box^{\triangle^{\boldsymbol{a}}} + \Box^{\triangle^{\boldsymbol{a-1}}}$.

Dickson used $\triangle$ to denote triangular numbers (though not $\Box$ to denote squares), but he does not use superscripts on $\triangle$, as Germain does. This notation, which could have been in use at that time or could have been Germain's invention, may not be immediately transparent. But if one reads $\triangle^a$ as T_a, the ath triangular number, and $\Box^a$ as a^2, the expressions make sense. Her use of superscripts where today we would use subscripts can be confusing, but her first equation is trivial and the second and third follow from her expanded derivations preceding them.

It is probably good that Germain's reputation does not depend on this slight observation, but it does perhaps fill a very small lacuna in the history of number theory problems of this type.

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Department of Mathematics and Computer Science, Santa Clara University, 500 El Camino Real, Santa Clara, California 95053-0290
E-mail address: galexand@math.scu.edu