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SELECTED MATHEMATICAL REVIEWS

of articles by John Milnor on the occasion of his award of the Abel Prize

MR0082103 (18,498d) 55.0X
Milnor, John
On manifolds homeomorphic to the 7-sphere.
Annals of Mathematics. Second Series **64** (1956), 399–405.

Until the appearance of this paper by Milnor it was thought probable that a given topological manifold might possess at most one differentiable structure. Now it is seen that a manifold as simple as the seven sphere S^7 possesses several inequivalent differentiable structures.

Milnor starts out by considering the class of compact oriented differentiable 7-manifolds M^7 such that the third and fourth cohomology groups with integer coefficients $H^3(M^7)$ and $H^4(M^7)$, are zero. For every such manifold he defines an invariant $\lambda(M^7)$ which is an integer modulo 7. He then defines several differentiable structures on the 7-sphere and proves that his invariant λ is not the same for all of these structures.

The definition of the invariant $\lambda(M^7)$ rests on Thom's theory of "cobordisme" [Comment. Math. Helv. **28** (1954), 17–86; MR0061823 (15,890a)], where it is proved that every compact 7-manifold M^7 is the boundary of an 8-manifold B^8. Let $\mu \in H_7(M^7)$ be an orientation, and $\nu \in H_8(B^8, M^7)$ an orientation such that $\partial\nu = \mu$. Define $\tau(B^8)$ to be the index of the quadratic form on $H^4(B^8, M^7)$ given by the formula $\alpha \to \langle \nu, \alpha \rangle$. Let $p_1 \in H^4(B^8)$ be the first Pontrjagin class, and $p_1 \in H^4(B^8, M^7)$ the unique class which maps into $p_1 (H^3(M^7) = H^4(M^7) = 0)$. Define a Pontrjagin number $q(B^8)$ to be $\langle \nu, \overline{p}_1{}^2 \rangle$.

Milnor's first theorem may now be stated. Theorem: The residue class of $2q(B^8) - \tau(B^8)$ modulo seven does not depend on the choice of the manifold B^8.

Having proved the preceding, he defines the invariant $\lambda(M^7)$ to be the residue class of $2q(B^8) - \tau(B^8)$, and it remains to consider examples of 7-manifolds. The examples in question are 3-sphere bundles over the 4-sphere with the rotation group SO(4) as structural group. Let R^4 be 4-dimensional euclidean space, and consider it as the space of quaternions. Define $f_{h,j} \colon S^3 \to \mathrm{SO}(4)$ by $f_{h,j}(u)v = u^h v u^j$, where the multiplication on the right is that of the quaternions. For each odd integer k, let $M_k{}^7$ be the total space of the 3-sphere bundle over S^4 determined by $f_{h,j}$, where $h + j = 1$, $h - j = k$.

The following theorem is now proved.

Theorem: The manifold $M_k{}^7$ is homeomorphic with the seven sphere, and the invariant $\lambda(M_k{}^7)$ is the residue class module 7 of k^2-1, and consequently if $k^2-1 \not\equiv 0 \bmod 7$ the manifold $M_k{}^7$ carries a differentiable structure which is not equivalent to the usual one on the sphere S^7.

It is also proved by the author that either (a) there exists a closed topological 8-manifold which does not possess any differentiable structure, or (b) the Pontrjagin class p, of an open 8-manifold is not a topological invariant.

J. C. Moore

MR0110107 (22 #990) 57.00

Milnor, John

Differentiable structures on spheres.

American Journal of Mathematics **81** (1959), 962–972.

In a previous paper [Ann. of Math. (2) **64** (1956), 399–405; MR0082103 (18,498d)] the author produced differentiable manifolds which are homeomorphic but not diffeomorphic to the 7-sphere S^7. Use was made of the fact that S^3 is parallelizable. Thus it was not clear a priori that similar examples could be produced in dimensions $\neq 7, 15$.

In this paper a general method of constructing differentiable manifolds which are topological spheres is exhibited. The method consists in matching $D^{m+1} \times S^n$ with $S^m \times D^{n+1}$ by a suitable diffeomorphism $f\colon S^m \times S^n \to S^m \times S^n$ of the boundaries. (D^q is the q-disk in euclidean q-space.) An important special case of such diffeomorphisms f is obtained from differentiable maps $f_1\colon S^m \to \mathrm{SO}_{n+1}$, $f_2\colon S^n \to \mathrm{SO}_{m+1}$ of spheres into rotation groups by setting $y' = f_1(x) \cdot y$ and $x = f_2(y') \cdot x'$, where $(x', y') = f(x, y)$. The resulting differentiable $(m+n+1)$-manifold is denoted by $M(f_1, f_2)$, and a simple sufficient condition on f_1, f_2 is given for $M(f_1, f_2)$ to be homeomorphic to S^{m+n+1}.

An invariant λ (of the J-equivalence class) of $M(f_1, f_2)$ is constructed and turns out to be of interest for dimensions of the form $m = 4r - 1$, $n = 4(k-r) - 1$. (Hence, $m + n + 1 = 4k - 1$.) The value $\lambda(M(f_1, f_2))$ is computed explicitly in terms of the Pontryagin classes of the $\mathrm{SO}_{4(k-r)}$-, resp. SO_{4r}-, bundles over S^{4r}, resp. $S^{4(k-r)}$, defined by f_1, resp. f_2, as characteristic maps. These Pontryagin classes in turn are known by a theorem of R. Bott [Bull. Amer. Math. Soc. **64** (1958), 87–89; MR0102804 (21 #1590)]. If $M(f_1, f_2)$ is diffeomorphic to S^{4k-1}, then $\lambda(M(f_1, f_2)) = 0$. Thus every $(4k-1)$-dimensional $M(f_1, f_2)$ homeomorphic to S^{4k-1} with a λ different from zero yields a non-standard differentiable structure on S^{4k-1}.

The problem of evaluating the possible values of λ (for a given k) is shown to amount to solving the following number-theoretic problem. Let $s_k = 2^{2k}(2^{2k-1} - 1)B_k/(2k)!$, where B_k is the kth Bernoulli number. Does there exist an integer r satisfying

$$k/3 < r \leq k/2,$$

such that the greatest odd factor $d_{k,r}$ in the denominator of

$$(2r-1)!(2(k-r)-1)!s_r s_{k-r}/s_k$$

is > 1? If such an r exists, the number $d_{k,r}$ is shown by the author to be a lower bound for the number of distinct differentiable structures on S^{4k-1}. The author has checked that the answer to the above question is affirmative for $k = 2$ and $4 \leq k \leq 14$. For instance, $d_{5,2} = 73$, showing that S^{19} has at least 73 distinct differentiable structures. Similarly, S^{31} is shown to possess at least $16,931,177$ distinct differentiable structures.

As usual the author's exposition is excellent.

{Reviewer's note: The above number-theoretic formulation is obtained by supplementing Lemma 5 of the paper under review with the statement on lines 9 and 8 from the bottom of page 348 of the paper by A. Borel and F. Hirzebruch, #988 above.}

M. A. Kervaire

MR0119209 (22 #9975) 55.00
Milnor, J.
On the cobordism ring Ω^* and a complex analogue. I.
American Journal of Mathematics **82** (1960), 505–521.

In this paper the author proves that the cobordism groups Ω^i defined by Thom [Comment. Math. Helv. **28** (1954), 17–86; MR0061823 (15,890a)] have no odd torsion. Thom proved that the cobordism groups Ω_i are isomorphic to the stable homotopy groups $\pi_{i+n}(M(SO_n))$ of certain "Thom spaces" $M(SO_n)$, and it is these stable homotopy groups which the author handles. Analogous Thom spaces $M(U_n)$ can be defined starting from the unitary group; the author's methods apply equally well to the stable homotopy groups $\pi_{i+2n}(M(U_n))$; we are promised that in part II of this paper these groups will be interpreted as "complex cobordism groups". For the moment he shows that they have no torsion and determines their structure.

The author handles his stable homotopy groups by using a spectral sequence due to the reviewer [ibid. **32** (1958), 180–214; Bull. Soc. Math. France **87** (1959), 277–280; MR0096219 (20 #2711); **22** #8500]. If X and Y are two finite CW-complexes, then this spectral sequence serves to relate the stable track groups $\{X, Y\}_n$ to the Ext groups $\mathrm{Ext}_A{}^{s,t}(H^*(Y; Z_p), H^*(X; Z_p))$. Here A denotes the mod p Steenrod algebra, and the cohomology groups H^* are regarded as modules over A. More generally, it is possible to replace Y by an object in a suitable category of stable objects. This is convenient here, because it enables one to replace the sequence of Thom spaces $M(SO_n)$ by a single "stable Thom object" $\mathbf{M}(SO)$. The cobordism group Ω^n is then isomorphic to the stable track group $\{S^0, \mathbf{M}(SO)\}_n$, and in principle this can be computed by the spectral sequence, taking X to be the sphere S^0.

In an earlier draft of this paper this computation was actually carried through. However, conclusions about the absence of torsion follow more easily from the following. Theorem 1: If $H^*(\mathbf{Y}; Z_p)$ is a free $A/(Q_0)$-module with even-dimensional generators, and if $C_*(\mathbf{Y}; Z)$ satisfies a finiteness condition, then the stable groups $\{S^0, \mathbf{Y}\}_n$ contain no p-torsion. In this theorem, (Q_0) denotes the two-sided ideal in A generated by the Bockstein boundary β_p. This theorem is proved very elegantly by taking X to consist of a circle with a 2-cell attached by a map of degree d. In this case everything in the spectral sequence is even-dimensional, and consequently $\{X, \mathbf{Y}\}_m$ is zero for m odd. On the other hand, if some $\{S^0, \mathbf{Y}\}_m$ contained p-torsion, then $\{X, \mathbf{Y}\}_n$ would have to be non-trivial for two consecutive values of n; this follows from the universal coefficient theorem for expressing $\{X, \mathbf{Y}\}_m$ in terms of $\{S^0, \mathbf{Y}\}_n$.

It remains, of course, to show that $H^*(\mathbf{M}(SO); Z_p)$ and $H^*(\mathbf{M}(U); Z_p)$ are free $A/(Q_0)$-modules with even-dimensional generators. This is done by adapting the argument which Thom used in the case $p = 2$ to show that $H^*(\mathbf{M}(O); Z_2)$ is a free A-module.

The paper is arranged as follows. After an introduction, § 1 contains necessary lemmas concerning the structure of the Steenrod algebra A, of $A/(Q_0)$, and of modules over these. These lemmas are based on the author's earlier work in Ann. of Math. (2) **67** (1958), 150–171 [MR0099653 (20 #6092)]. § 2 is about stable objects, the spectral sequence, and Theorem 1. § 3 treats the particular stable object $\mathbf{M}(U)$, its cohomology and stable homotopy groups. § 4 indicates how

$\mathbf{M}(SO)$ can be treated by trivial amendments of the details in § 3. The paper concludes with applications and suggestions for further research.

One regrets that this significant paper could not have appeared earlier, in some form; but one can only applaud the elegance and clarity of the final version.

J. F. Adams

MR0148075 (26 #5584) 57.10

Milnor, John W.

Groups of homotopy spheres. I.

Annals of Mathematics. Second Series **77** (1963), 504–537.

The authors aim to study the set of h-cobordism classes of smooth homotopy n-spheres; they call this set Θ_n. They remark that for $n \neq 3, 4$ the set Θ_n can also be described as the set of diffeomorphism classes of differentiable structures on S^n; but this observation rests on the "higher-dimensional Poincaré conjecture" plus work of Smale [Amer. J. Math. **84** (1962), 387–399], and it does not really form part of the logical structure of the paper. The authors show (Theorem 1.1) that Θ_n is an abelian group under the connected sum operation. (In § 2, the authors give a careful treatment of the connected sum and of the lemmas necessary to prove Theorem 1.1.)

The main task of the present paper, Part I, is to set up methods for use in Part II, and to prove that for $n \neq 3$ the group Θ_n is finite (Theorem 1.2). (For $n = 3$ the authors' methods break down; but the Poincaré conjecture for $n = 3$ would imply that $\Theta_3 = 0$.) We are promised more detailed information about the groups Θ_n in Part II.

The authors' method depends on introducing a subgroup $bP_{n+1} \subset \Theta_n$; a smooth homotopy n-sphere qualifies for bP_{n+1} if it is the boundary of a parallelizable manifold. The authors prove in § 4 that the quotient group Θ_n/bP_{n+1} is finite (Theorem 4.1). More precisely, they prove that bP_{n+1} is the kernel of a homomorphism $p'\colon \Theta_n \to \Pi_n/\mathrm{Im}J$, where Π_n is the stable group $\pi_{n+k}(S^k)$ and $\mathrm{Im}\,J$ is the image of the classical J-homomorphism. § 4 ends by giving (explicitly) the groups Θ_n/bP_{n+1} for $n \leq 8$ and the groups bP_{n+1} for $n \leq 19$, referring the reader to Part II for details.

The proof given in § 4 depends on results in § 3. In this section, Theorem 3.1 states that every homotopy sphere is S-parallelizable, that is, its tangent bundle is stably trivial. The proof uses previous work of the same authors, and involves quoting information about the J-homomorphism. The remaining lemmas in § 3 concern the stability of bundles.

It remains to prove that the groups bP_{n+1} are finite. The authors divide two cases. If n is even they prove that the groups bP_{n+1} are zero. That is, in §§ 5, 6 they prove (Theorem 5.1): If a smooth homotopy sphere of dimension $2k$ bounds an S-parallelizable manifold M, then it bounds a contractible manifold. The proof consists of simplifying M by surgery [J. Milnor, Proc. Sympos. Pure Math., Vol. III, pp. 39–55, Amer. Math. Soc., Providence, R.I., 1961; MR0130696 (24 #A556)]. The details are technical, and appear to be comparable with work of C. T. C. Wall, which also results in a proof of the same theorem [Trans. Amer. Math. Soc. **103** (1962), 421–433; MR0139185 (25 #2621)]. § 5 completes the proof for k even; the case in which k is odd is treated in § 6. Here the authors introduce the notion of a "framed manifold", that is, a smooth manifold M plus a given trivialisation of the stable tangent bundle of M. The authors arrange to carry this extra structure through the technique of surgery, making use of it as they go.

The case in which n is odd is treated in §§ 7, 8. It is shown that the groups bP_{2k} are finite cyclic, and for k odd they are either 0 or Z_2 (Corollary 7.6; Theorem 8.5). The case in which k is even is dealt with in § 7. Here the only obstruction to performing surgery on M is the signature or index $\sigma(M)$ (Lemma 7.3). This leads to the following result (Theorem 7.5). Let Σ_1 and Σ_2 be homotopy spheres of dimension $4m-1$ ($m>1$) which bound S-parallelizable manifolds M_1 and M_2. Then Σ_1 and Σ_2 and h-cobordant if and only if $\sigma(M_1)\equiv\sigma(M_2) \bmod \sigma_m$. Here σ_m is a certain positive integer. § 7 concludes by giving explicit information about the integer σ_m and the order of the groups bP_{4m} and Θ_{4m-1}. The reader is referred to Part II for details.

The cases $k=1,3,7$ are exceptional; the group bP_{2k} is then zero (Lemma 7.2). The case "k odd$\neq 1,3,7$" is studied in § 8. In this case the only obstruction to performing surgery on M is an "Arf invariant" lying in Z_2. The authors conjecture that in this case the group bP_{2k} is always Z_2 rather than 0; but this is known only for $k=5,9$.

J. F. Adams

MR0161345 (28 #4553a) 57.30; 57.20

Milnor, J.

Topological manifolds and smooth manifolds.

Proc. Internat. Congr. Mathematicians (Stockholm, 1962), 132–138, *Inst. Mittag-Leffler, Djursholm*, 1964.

MR0161346 (28 #4553b) 57.30; 57.20

Milnor, J.

Microbundles. I.

Topology. An International Journal of Mathematics **3** (1964), 53–80.

In this paper the author proves the results stated in his lecture to the International Congress of Mathematicians, 1962 [#4553a]. We refer to this lecture as [L].

The notion of a "microbundle" (§§ 1, 2) is essentially obtained from that of a vector bundle by (i) restricting attention to a neighbourhood of the zero cross-section, and (ii) abandoning all conditions of "linearity", so that one uses only topological conditions. This notion is introduced so that one may assign (§ 2) to each topological manifold a tangent microbundle, analogous to the tangent bundle of a smooth manifold. Suppose given a smooth manifold M; then on the one hand we can first take its tangent vector bundle ξ, and then take the microbundle underlying ξ; on the other hand we can first take the topological manifold $|M|$ underlying M, and then take the tangent microbundle of $|M|$. It is proved that the two results agree (Theorem 2.2 = Theorem 1 of [L]).

Much of the theory of vector bundles carries over to microbundles. Thus, we have induced microbundles (§ 3) and Whitney sums (§ 4). It is stated in § 3 that if two maps are homotopic, then the corresponding induced microbundles are isomorphic (Theorem 3.1 = Theorem 3 of [L]). The proof, which requires some work, is given in § 6.

One can introduce a factor k_{Top} based on microbundles, analogous to the factor $k_O = \tilde{K}$ based on vector bundles, and one has a natural transformation $k_O \to k_{\mathrm{Top}}$ (§ 4; Corollary 4.3 = Theorem 4 of [L]). For this purpose the key result (Theorem 4.1) is that any microbundle x over a finite-dimensional simplicial complex B admits an "inverse" y such that the Whitney sum $x+y$ is trivial. The proof, which requires some work, is given in §§ 4, 7.

Next, one tries to introduce normal microbundles. If we have topological manifolds $M \subset N$, then in general M does not have a normal microbundle. However, M has a normal microbundle in $N \times R^q$ for sufficiently large q (Theorem 5.8). The proof takes some work (§ 5). A normal bundle n is related in the usual way to the two tangent bundles: $t_M + n \cong t_N$.

This leads to results on the smoothing problem. Theorem 5.12, which corresponds to Theorem 2 of [L], is sharpened to give Theorem 5.13: Let ξ be a vector bundle over the topological manifold M; then some product $M \times R^q$ can be smoothed so as to have tangent bundle isomorphic to ξ plus a trivial bundle if and only if the homomorphism $k_O(M) \to k_{\mathrm{Top}}(M)$ carries the class of ξ to the class of the tangent microbundle t_M.

It is therefore interesting to examine the difference between $k_O(X)$ and $k_{\mathrm{Top}}(X)$. The fact is that the natural transformation $k_O(X) \to k_{\mathrm{Top}}(X)$ need be neither mono nor epi (Lemmas 9.1 and 9.4 = Theorem 5 of [L]). The key result (Theorem 8.1) states that the image in $k_{\mathrm{Top}}(S^{4n})$ of the generator in $k_O(S^{4n})$ is divisible by a certain integer (whose definition involves the Bernoulli numbers). Heavy hammers are needed for the proof (§ 8); the author uses the methods of Milnor and Kervaire ["Groups of homotopy spheres", II, in preparation] and Wall [Ann. of Math. (2) **75** (1962), 163–189; MR0145540 (26 #3071)]; he also quotes results from Hirsch, from Smale and from the reviewer.

Finally, the following results are proved in § 9 by exploiting the fact that k_O and k_{Top} are different. (i) The tangent vector bundle of a manifold is not a topological invariant (Theorem 9.2 = Corollary 1 of [L]). (ii) The Pontryagin classes of an open manifold are not topological invariants (Corollary 9.3). (iii) There exists a topological manifold M such that no product $M \times M'$ can be smoothed (Theorem 9.5 = Corollary 2 of [L]).

This paper is clearly required reading for any worker in the field.

J. F. Adams

MR0196736 (33 #4922) 55.40; 55.25

Milnor, J.

Whitehead torsion.

Bulletin of the American Mathematical Society **72** (1966), 358–426.

This excellent reworking and exposition will earn the gratitude of many topologists.

The author begins with a brief historical introduction to "torsion" in the sense of Reidemeister, Franz and de Rham, and in the sense of J. H. C. Whitehead. The actual work begins with algebra. In § 1 the group $K_1(A)$ is defined for any associative ring A with unit [following Bass, same Bull. **70** (1964), 429–433; MR0160825 (28 #4035); Inst. Hautes Études Sci. Publ. Math. No. 22 (1964), 5–60; MR0174604 (30 #4805)]. The author also gives examples and a survey of theorems. This sort of algebra continues in § 6 and two appendices. In § 6 the Whitehead group $\mathrm{Wh}(\pi)$ is defined for any multiplicative group π, as a quotient of $K_1(Z\pi)$. Again, the author gives examples and a survey of theorems. The first appendix shows the relationship between the groups $K_1(A)$ and the "congruence subgroup theorem", that is, the proposition proved for the case $A = Z$ by Bass, Lazard and Serre [ibid. **70** (1964), 385–392; MR0161913 (28 #5117)]. The second appendix is intended to motivate the notation $K_1(A)$; it introduces the groups $K_0(A)$ and gives the usual examples, with a survey of theorems.

The object of §§ 2–5 is to define the torsion invariant for suitable chain complexes. Suppose given a chain complex C of A-modules such that each chain group C_i and each homology group H_i is A-free with a given base; suppose, also, that each group of boundaries B_i is free. Then, in § 3, the torsion $\tau(C)$ of C is defined; it is an element of $\overline{K}_1(A)$. (To define $\overline{K}_1(A)$, observe that the element -1 in A gives an element of order 2 in $K_1(A)$; then $\overline{K}_1(A)$ is the corresponding quotient of $K_1(A)$.) The definition requires some elementary constructions with bases in free modules, and these are provided in § 2. § 3 also contains the following property of $\tau(C)$: If $0 \to C' \to C \to C'' \to 0$ is an exact sequence of chain complexes (as above), then $\tau(C) = \tau(C') + \tau(C'') + \tau(H)$, where H is the usual exact homology sequence, considered as a chain complex.

The object of § 4 is to remove the assumption that each group of boundaries B_i is free. This is done by observing that B_i is at least "stably free". (A module M is "stably free" if $M \oplus F$ is free for some F.) The notion of "stable base" is introduced, and it is observed that the construction and results of § 2 generalise to this case. Hence, the constructions and results of § 4 also generalise.

The object of § 5 is to prove an algebraic theorem which can later be applied to show that torsion is invariant under subdivision of a complex. The situation is the analogue, for torsion, of the computation of homology from "simplex blocks".

At this point, the author passes from algebra to geometry. In § 7 he defines the Whitehead torsion $\tau(f)$ whenever $f\colon X \to Y$ is a homotopy equivalence between finite CW-complexes. He also obtains properties of the torsion, for example, $\tau(gf) = \tau(g) + g^*\tau(f)$. The case of a general equivalence $f\colon X \to Y$ is handled by reducing it to the case in which f is an inclusion map; so, one has to consider a CW-pair K, L. One takes the universal covering pair $\hat{K}, \hat{L}$ and applies §§ 3, 4 to its chain complex. (The ring A becomes the group ring of the fundamental group π.) It is proved that the resulting torsion $\tau(K, L)$ is invariant under subdivision of (K, L).

In § 8 the author discusses variants of the definition. First return to the algebraic context, and suppose given a homomorphism $h\colon A \to A'$ of rings and a chain complex C over A. Then one can form a chain complex $C' = A' \otimes C$ over A'. It may happen that C' is acyclic, although C is not. In this case one can define the torsion $\tau(C')$ starting from a preferred base in C, without needing any preferred base in homology. When this remark is applied to the topological context, it enables one to dispense with the assumption that $i\colon L \to K$ is a homotopy equivalence, at the price of introducing h. In the first variant of the method, h is a one-dimensional complex representation of π. This yields the original Reidemeister-Franz torsion; examples are given. In the second variant, h is an n-dimensional real representation of π. A third variant is given in § 12.

In § 9 the author moves on to smooth manifolds, and defines the torsion $\tau(W, M)$ of an h-cobordism W between M and M'. In § 10 he quotes the thoerem that in dimensions ≥ 6, such an h-cobordism is a product if and only if $\tau(W, M) = 0$. This motivates the duality formula relating $\tau(W, M)$ and $\tau(W, M')$, which is then proved. In § 11 he quotes and reproves the theorem that in dimensions ≥ 6, any element τ in $\mathrm{Wh}(\pi_1(M))$ can be realised as the torsion of some h-cobordism. It follows that h-cobordisms of dimension ≥ 6 are classified by their torsion (11.3). Applications are given.

Finally, in § 12 the author gives an extensive survey of results on lens spaces.

J. F. Adams