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Classifying spaces of sporadic groups, by David J. Benson and Stephen D. Smith, Mathematical Surveys and Monographs 147, American Mathematical Society, Providence, RI, 2008, xvi + 289 pp., ISBN 978-0-8218-4474-8

The use of geometry to study and explain the properties of finite groups has a long history, going back at least to the construction by Tits in the 1950s of geometries associated to Chevalley groups. The book being reviewed [BS] is centered around the connections between the sporadic simple groups and associated geometric structures, with a particular focus on applying such structures to compute the cohomology of those groups.

1. **Incidence geometries.** The idea of an incidence geometry, and of associating such geometries to simple groups, was first formulated by Tits in the 1950's [Ti]. Fix a finite set I. A *geometry over* I is a triple $(\Gamma,\tau,*)$, where Γ is a set (the objects in the geometry), $\tau\colon \Gamma \longrightarrow I$ is a map, and $*$ is a symmetric, reflexive *incidence relation* on Γ. The only condition these must satisfy is that for $x,y\in\Gamma$, $\tau(x)=\tau(y)$ and $x*y$ imply $x=y$. One thinks of I as the set of *types* of objects in the geometry, and the condition says that two distinct objects of the same type cannot be incident.

A *flag* in a geometry $(\Gamma,\tau,*)$ over I is a set of pairwise incident objects. For $\varnothing\neq J\subseteq I$, a *flag of type* J is a flag which contains one object of type j for each $j\in J$.

We give two of the motivating examples of geometries. First, fix a field K and a finite dimensional K-vector space V with $\dim(V)=n<\infty$. Set $I=\{1,2,\dots,n-1\}$. Let Γ be the set of all proper nonzero vector subspaces of V, and define $\tau\colon\Gamma\longrightarrow I$ by setting $\tau(W)=\dim(W)$. For a pair of subspaces $W,X\subseteq V$, $W*X$ if $W\subseteq X$ or $X\subseteq W$.

Thus the objects in the geometry are the (linear) lines, planes, 3-spaces, etc., in V, or equivalently, the points, lines, planes, etc., in the projective space $P(V)$. A flag is a sequence of subspaces each contained in the next. The projective linear group $P\operatorname{Aut}_K(V)\cong PGL_n(K)$ acts on this geometry by acting on the set of objects while preserving type and incidence.

As a second example, let G be any group, and fix a set of nontrivial proper subgroups $\{G_1,\dots,G_k\}$, no two of which are G-conjugate. To make it more interesting, we also assume that for $i\neq j$, G_i is not G-conjugate to a subgroup of G_j. Let Γ be the disjoint union of the sets G/G_i of left cosets gG_i. Set $I=\{1,\dots,k\}$ and $\tau(gG_i)=i$. Two objects (cosets) $X,Y\in\Gamma$ are incident if $X\cap Y\neq\varnothing$. The group G acts on this geometry via the usual left action on each G/G_i. A flag in the geometry is a set of cosets any two of which have nonempty intersection.

In fact, the first example is (isomorphic to) a special case of the second. For any given V as above, fix a *maximal* flag $0\subsetneq V_1\subsetneq V_2\subsetneq\cdots\subsetneq V_{n-1}\subsetneq V$, where $\dim(V_i)=i$. Set $G=P\operatorname{Aut}_K(V)\cong PGL_n(K)$, and let $G_i\leq G$ be the subgroup of all classes of automorphisms which send V_i to itself (not necessarily by the identity). Then for $g\in G$, gG_i is the set of all elements of G which send

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V_i to $g(V_i)$ (where automorphisms are composed from right to left). This defines a bijection between the nontrivial proper subspaces V and $\coprod_{i=1}^{k} G/G_i$: a bijection which clearly preserves types and which also preserves the incidence relations.

2. **Geometries for finite simple groups.** By a *geometry for a group G*, we mean a geometry Γ upon which G acts by permuting its objects, and in a way which preserves types and incidence. The geometry is *p-local* if the stabilizer subgroups of the objects in Γ are all p-local subgroups of G; i.e., normalizers of nontrivial p-subgroups of G. Sometimes one also requires that G act transitively on the set of all flags of a given type J.

To each geometry Γ, one associates a simplicial complex $|\Gamma|$—the *geometric realization of* Γ—whose vertices are the objects of Γ, and where a set of vertices spans a simplex if and only if the corresponding objects are pairwise incident. As will be described below, the properties of this space $|\Gamma|$ (such as connectivity, homology, fundamental group) play an important role when using the geometry to get information about the group.

For example, when G is a finite group of Lie type in characteristic p, then the *Tits building* for G is associated to a natural p-local geometry for G. The objects of the geometry are the maximal parabolic subgroups—the maximal subgroups which contain Borel subgroups—where two such subgroups are incident if their intersection contains a Borel subgroup. The flags in the geometry correspond to the simplices in the building.

As one example of how this is used, we mention the Solomon–Tits theorem. In the form shown in [So, Theorems 1–2], this says that the building (or geometric realization of the geometry) associated to a group G with BN pair of rank ≥ 2 has a unique nonzero reduced homology group (over $\mathbb{Z}$), and that after tensoring with $\mathbb{Q}$, this is isomorphic to the Steinberg representation for G.

The systematic study of geometries for sporadic simple groups was begun by Buekenhout [Bu], and continued by Ronan and Smith [RS] and Ivanov [Iv1], among others. Ronan and Smith gave examples of 2-local geometries for twelve of the groups, including M_{22}, M_{24}, the Conway groups Co_2 and Co_1, and the monster and baby monster. These authors also showed how *Dynkin diagrams* can be associated to geometries for sporadic groups, by analogy with those for Lie groups.

Geometries of certain sporadic groups played an important role in constructing computer-free uniqueness proofs of several of the sporadic groups. A general framework for doing this was set up by Aschbacher and Segev [AS1, AS2]. Their general result (cf. [AS2, Theorem 1]) is stated in terms of graphs, but in the applications (cf. [AS2, § 7–8], the graphs they use are mostly associated to geometries. Part of their procedure involves the fundamental group of the geometric realization of this graph or geometry; for example, showing it is simply connected. (See, e.g., Theorem 1 and Section 8 in [AS2]. A graph is triangulable in their terminology if its clique complex—the largest simplicial complex with the same 1-skeleton—is simply connected.)

The geometries studied by Benson and Smith [BS] all have the following form. Let I be a finite set of indices, fix $H_i \leq G$ for each $i \in I$, and set $H_J = \bigcap_{j\in J} H_j$ for each $\varnothing \neq J \subseteq I$. Let $\Delta_{\mathcal{H}_I}$ be the geometry whose objects are the subgroups G-conjugate to some H_i, and where a subgroup of type i (i.e., conjugate to H_i) is incident to a subgroup of type j if their intersection is G-conjugate to H_{ij}. In all cases, they also assume that the geometry is *flag transitive*: if $J \subseteq I$, K_j is

G-conjugate to H_j for each $j \in J$, and the K_j are pairwise incident, then there is $g \in G$ such that $gK_jg^{-1} = H_j$ for each $j \in J$. (So in particular, $\bigcap_{j\in J} K_j$ is G-conjugate to H_J in this case.)

In some cases, a geometry for G can be described in terms of structures on which G acts. For example, a geometry for the Matthieu group M_{24} can be defined whose objects consist of the octads, trios, and sextets in the Steiner system for G, with incidence defined in a natural way.

3. **Posets of p-subgroups of a group.** Let $\mathcal{S}_p(G)$ be the poset of all nontrivial p-subgroups of a given finite group G. Let $|\mathcal{S}_p(G)|$ be its geometric realization: the simplicial complex with one vertex for each element $1 \neq P \leq G$ in $\mathcal{S}_p(G)$, one edge for each pair $P < Q$ of elements (connecting the vertices corresponding to P and Q), and more generally one n-simplex for each strictly increasing sequence $P_0 < P_1 < \cdots < P_n$ of elements in $\mathcal{S}_p(G)$.

This poset, and the homotopy properties of its geometric realization, was first investigated in detail by Quillen [Q]. He showed that if $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ denotes the subposet of nontrivial elementary abelian p-subgroups of G, then $|\mathcal{A}_p(G)|$ is a strong deformation retract of $|\mathcal{S}_p(G)|$. He also showed [Q, § 3] that when G is a Chevalley group over a finite field of characteristic p, then $|\mathcal{A}_p(G)| \simeq |\mathcal{S}_p(G)|$ has the homotopy type of the Tits building for G, thus providing a link (at least in this special case) between the poset $\mathcal{S}_p(G)$ and the p-local geometry for G which had already been studied.

Among other things, Quillen also proved that $|\mathcal{A}_p(G)|$ is contractible if $O_p(G) \neq 1$; i.e., if G has a nontrivial normal p-subgroup. He also conjectured that the converse should hold [Q, Conjecture 2.9], and proved his conjecture when G is solvable. This conjecture has since been proven in many other cases, but it is still unknown whether it holds in general.

We now list some of the other posets which play an important role in the discussion which follows. A p-subgroup P of a finite group G is *radical* if $O_p(N_G(P)/P) = 1$, and it is p-centric if $Z(P) \in \mathrm{Syl}_p(C_G(P))$ (equivalently, $C_G(P) = Z(P) \times C'_G(P)$ for some $C'_G(P)$ of order prime to p). Let $\mathcal{B}_p(G)$ and $Ce_p(G)$ denote the sets of nontrivial radical p-subgroups, and p-centric subgroups, respectively, and set $\mathcal{B}_p^{cen}(G) = \mathcal{B}_p(G) \cap Ce_p(G)$. The inclusion of $|\mathcal{B}_p(G)|$ into $|\mathcal{S}_p(G)|$ was shown by Bouc [Bc] to be a homotopy equivalence.

One more family of p-subgroups plays an important role in [BS]. Let $\mathcal{E}_0$ denote the smallest set of elements of order p in G such that

- $\mathcal{E}_0$ is closed under G-conjugacy and contains all central elements of order p in each Sylow p-subgroup of G; and
- the product of any two commuting elements of $\mathcal{E}_0$ lies in $\mathcal{E}_0 \cup \{1\}$.

Let $\mathcal{E}_p(G) \subseteq \mathcal{A}_p(G)$ be the set of elementary abelian p-subgroups all of whose elements lie in $\mathcal{E}_0 \cup \{1\}$.

4. **Decompositions of group cohomology and of classifying spaces.** The use of geometries and posets of p-subgroups for constructing decompositions of group cohomology $H^*(G; \mathbb{F}_p)$ is most easily explained with reference to a classifying space for G.

A *classifying space* of a discrete group G is a space BG whose fundamental group is isomorphic to G, and whose universal covering space is contractible. Usually, one also requires that the space be "nice" in some sense; e.g., a cell complex.

More explicitly, one can construct "the" classifying space BG to be the geometric realization of the category which has one object, with endomorphism group G. It is thus a cell complex with one vertex, one edge for each element of G, one 2-simplex (triangle) for each pair (g,h) of elements of G (a triangle with edges corresponding to g, h, and hg), etc. We refer to [Be, § 2.2, 2.4] for a more detailed (but still succinct) description of these spaces (and note that BG is an Eilenberg–MacLane space of type $K(G,1)$).

Let EG denote the universal covering space of a classifying space BG. The *deck transformations* of EG—the group of self maps which cover the identity on BG—define a free action of G on EG, which for convenience we will assume to be a right action. (Any action on the left can be regarded as a right action by replacing the left action of g by a right action of g^{-1}.)

One of the key properties of a classifying space BG is that for any abelian group M of coefficients (in fact, for any $\mathbb{Z}[G]$-module M), $H^*(BG;M) \cong H^*(G;M)$. This is seen upon noting that the homology chain complex $C_*(EG;\mathbb{Z})$, with the linear action of G induced by its action on EG, is a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z} = H_0(EG;\mathbb{Z})$. In other words, the cohomology of the topological space BG is isomorphic to the algebraically defined cohomology of the group G. We again refer to Benson's book [Be, Theorem 2.2.3] for more detail.

When X is any topological space with (continuous) left G-action, the *Borel construction* for X is the space

$$EG \times_G X \stackrel{\text{def}}{=} \big(EG \times X\big)/\!\sim,$$

where $(ag,x) = (a,gx)$ for each $a \in EG$, $g \in G$, and $x \in X$. For example, when $X = *$ is a point, then $EG \times_G * \cong EG/G \cong BG$.

Assume $X = |\Delta|$ for some simplicial complex Δ upon which G acts simplicially, and with the additional assumption that the vertices of each simplex lie in distinct orbits. For each $i \geq 0$, let Δ_i be the set of i-simplices in Δ, and for each simplex $\sigma \in \Delta_i$, let $G_\sigma \leq G$ be its stabilizer under the G-action. Then for any p, the filtration of $H^*(EG \times_G |\Delta|;\mathbb{F}_p)$ via the skeleta of Δ defines a spectral sequence

$$E_1^{ij} = \prod_{[\sigma]\in\Delta_i/G} H^j(G_\sigma;\mathbb{F}_p) \implies H^{i+j}(EG \times_G |\Delta|;\mathbb{F}_p). \tag{1}$$

For a fixed prime p, a G-simplicial complex Δ is defined to be *ample* if the natural map $EG \times_G |\Delta| \longrightarrow BG$, induced by sending $|\Delta|$ to a point, induces an isomorphism in mod p (co)homology. Thus the spectral sequence (1) converges to $H^*(BG;\mathbb{F}_p) \cong H^*(G;\mathbb{F}_p)$ in this case. When Δ is ample, Δ is called *normalizer sharp* if in addition, the spectral sequence (1) collapses in the sense that $E_2^{ij} = 0$ for all $i > 0$ and all j.

When Δ is normalizer sharp and $\dim(\Delta) = n < \infty$, then for each j, there is an exact sequence

$$0 \longrightarrow H^j(G;\mathbb{F}_p) \longrightarrow E_1^{0j} \xrightarrow{d_1^{0j}} E_1^{1j} \xrightarrow{d_1^{1j}} \cdots \xrightarrow{d_1^{n-1,j}} E_1^{nj} \longrightarrow 0\ ,$$

where the d_1^{ij} are differentials in the spectral sequence (1). This can be expressed by saying that

$$H^*(G;\mathbb{F}_p) \cong \bigoplus_{\sigma\in\Delta/G} (-1)^{\dim(\sigma)-1} H^*(G_\sigma;\mathbb{F}_p). \tag{2}$$

Formula (2) can be regarded simply as a tool for computing $\dim_{\mathbb{F}_p}(H^j(G;\mathbb{F}_p))$ for each j, but there are also ways of making it more precise.

When Δ is the geometric realization of a poset $\mathcal{C}$ of p-subgroups of G (with G-action defined by conjugation), then $\mathcal{C}$ is called ample or normalizer sharp if Δ is. (The term "normalizer sharp" is used because in that case, the stabilizer subgroups of vertices are the normalizers of the corresponding subgroups.) For example, the posets $\mathcal{S}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{B}_p^{cen}(G)$, and $Ce_p(G)$ are all ample by results of Dwyer [Dw1, 1.16–1.18]. They are all normalizer sharp: this holds for $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ by a result of Webb [Wb, Corollary 2.5.1] (together with [Bc] and the fact that nontrivial p-subgroups have a contractible fixed point set on $|\mathcal{S}_p(G)|$); and for all four of the posets by Grodal's theorem [Gr, Theorem 7.3]. The poset $\mathcal{E}_p(G)$ is ample and normalizer sharp: the sharpness was shown in [GS, Theorem 1.2].

From the topological point of view, when Δ is ample, the mod p equivalence of $EG \times_G |\Delta|$ with BG leads to a *p-local* decomposition of BG as a *homotopy colimit* of the spaces $EG \times_G (G/G_\sigma) \simeq BG_\sigma$ for simplices $\sigma \in \Delta_0$. (See, e.g., [BS, § 4.5–7] for more on decompositions via homotopy colimits.) It was in the context of decompositions of classifying spaces with respect to certain families of subgroups that Dwyer [Dw1, Dw2] first used the terms "ample" and "sharp", to systematize the different ways of decomposing a classifying space. In addition to the "normalizer decomposition", Dwyer also defined a subgroup decomposition and a centralizer decomposition for any ample family of subgroups, and corresponding notions of sharpness. See also [GS] for a later survey of the different types of decompositions and the corresponding ampleness and sharpness properties.

5. **Contents of the book.** The main result in this book is the construction of an explicit 2-local geometry $\Delta_{\mathcal{H}_I}$ for each sporadic group G, which is shown to be normalizer sharp by comparison to some poset of p-subgroups of G. In many cases, lists of radical p-subgroups made by various authors are used to help compare the chosen geometry for G with $\mathcal{B}_2^{cen}(G)$. This then leads, via (2), to an additive decomposition of $H^*(G;\mathbb{F}_2)$ as an alternating sum of the cohomology groups $H^*(H_J;\mathbb{F}_2)$ for $\varnothing \neq J \subseteq I$. It also leads to a 2-local decomposition of BG itself as a homotopy colimit of the classifying spaces BH_J for subsets $\varnothing \neq J \subseteq I$.

Many of these decompositions were already known. The importance of this work is that the authors carry out this procedure systematically, for all of the sporadic simple groups.

To illustrate this better, we sketch their results for two of the groups. When $G = M_{12}$, they fix two subgroups $H_1 \cong Q_8 \rtimes \mathrm{Aut}(Q_8)$ $(2_+^{1+4} : \Sigma_3)$, and $H_2 \cong C_4^2 \rtimes D_{12}$. These are chosen so that $H_1 = N_G(A_1)$ and $H_2 = N_G(A_2)$, where $A_1 \leq A_2$ and $A_i \cong C_2^i$; this determines the H_i as a pair up to G-conjugacy. Thus $I = \{1,2\}$, and the geometry they study for M_{12} is $\Delta_{\mathcal{H}_I}$. They then consider the poset $\mathcal{E}_2^- \subseteq \mathcal{E}_2(G)$ containing only the subgroups G-conjugate to A_1 or A_2, and show that the inclusion of $|\mathcal{E}_2^-|$ in $|\mathcal{E}_2|$ is a homotopy equivalence. Thus by [BS, 5.8.11] (but see the remarks below), $\mathcal{E}_2^- \cong \Delta_{\mathcal{H}_I}$ is normalizer sharp since $\mathcal{E}_2(G)$ is. This leads to an exact sequence

$$0 \longrightarrow H^*(G;\mathbb{F}_2) \longrightarrow H^*(H_1;\mathbb{F}_2) \oplus H^*(H_2;\mathbb{F}_2) \longrightarrow H^*(H_{12};\mathbb{F}_2) \longrightarrow 0$$

and to their description of $H^*(G;\mathbb{F}_2)$ as a formal difference of the other two terms.

As a second example, we take the case $G = Co_1$, with $I = \{1,2,4,11\}$, and $H_i = N_G(A_i)$ for a certain choice of elementary abelian 2-subgroups

$A_1 < A_2 < A_4 < A_{11}$ with $\mathrm{rk}(A_i) = i$. The resulting geometry $\Delta_{\mathcal{H}_I}$ is that already studied in [RS]. The radical 2-subgroups of G were listed by Sawabe [Sw1], who showed that there are 27 conjugacy classes in $\mathcal{B}_2^{cen}(G)$. Sawabe had also showed (in [Sw2]) that if $\mathcal{B}_2^{cen-} \subseteq \mathcal{B}_2^{cen}(G)$ denotes the subposet of those conjugate to one of the 15 subgroups $O_2(H_J)$ for $\varnothing \neq J \subseteq I$, then the inclusion $|\mathcal{B}_2^{cen-}| \subseteq |\mathcal{B}_2^{cen}(G)|$ is a homotopy equivalence. Thus $\Delta_{\mathcal{H}_I}$ and $\mathcal{B}_2^{cen-}$ are both normalizer sharp, giving a rank four decomposition of $H^*(G; \mathbb{F}_2)$.

The other examples are mostly similar in style. But many of them contain subtleties which are impossible to describe in a brief description such as this one.

This book provides a very good introduction for group theorists to some structures in homotopy theory, such as classifying spaces and p-completion. It also provides a good introduction for homotopy theorists to the geometries of finite simple groups and to some of the properties of sporadic simple groups. For both groups of researchers, it will be an excellent reference for many of the properties of the sporadic simple groups.

We note that the results 5.1.16, 5.1.17, 5.6.7, 5.8.4 in the book are incorrect as stated. One way to fix them is to add the additional hypothesis in each case that the orbit space $|\Delta|/G$ be contractible (or that it have the mod p-cohomology of a point). Alternatively, one can add the assumption that the stabilizer of each point in $|\Delta|$ have order a multiple of p. The error(s) arose because the results of Webb which they cite are stated by him in terms of Tate cohomology, rather than ordinary cohomology—and are not true in degree zero.

Theorem 5.8.11 requires the additional assumption that the complexes Δ and Δ' be finite dimensional. Even in this form, we do not know any published reference, although it does follow from Smith theory and Webb's result ([BS, Theorem 5.1.16] corrected).

Tables 1–3 give a very brief (in many cases oversimplified) overview of the methods used to handle each group. We include it only as a guide for readers who want to choose a few specific examples to look at. For each G, the second line gives the rank of the geometry used (i.e., the order of I), and the third line gives the poset(s) of p-subgroups with which it was compared. A superscript "$-$" means that they use an explicitly defined subset of the standard poset; e.g., $\mathcal{B}_2^{cen-}$ is a (G-invariant) subset of $\mathcal{B}_2^{cen}(G)$.

TABLE 1

G	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	J_1	J_2	J_3	J_4
rk	2	2	3	3	3	1	2	3$*$	4
poset	$\mathcal{A}_2$	$\mathcal{E}_2^-$, $\mathcal{B}_2^{cen}$	$\mathcal{A}_2$	$\mathcal{A}_2$	$\mathcal{A}_2$	$\mathcal{A}_2$	$\mathcal{E}_2$, $\mathcal{B}_2^{cen}$	$\mathcal{A}_2^-$	$\mathcal{B}_2^-$

TABLE 2

G	Co_3	Co_2	Co_1	HS	McL	Suz	He	Ru	$O'N$
rk	3	4	4	3	3	3	3	3	2
poset	$\mathcal{E}_2^-$	$\mathcal{B}_2 = \mathcal{B}_2^{cen}$	$\mathcal{B}_2^{cen-}$	$\mathcal{E}_2^-$	$\mathcal{A}_2^-$	$\mathcal{E}_2^-$, $\mathcal{B}_2^{cen}$	$\mathcal{B}_2^{cen-}$	$\mathcal{E}_2^-$, $\mathcal{B}_2^{cen}$	$[\mathcal{B}_2^{cen}]$

Table 3

G	Ly	Fi_{22}	Fi_{23}	Fi'_{24}	$F_5 = HN$	$F_3 = Th$	$F_2 = B$	$F_1 = M$
rk	4	4	4	5	3	2	5	5
poset	$\mathcal{A}_2^-$	$\mathcal{B}_2^{cen}$	$\mathcal{B}_2^{cen}$	$\mathcal{B}_2^{cen}$	$\mathcal{B}_2^{cen-}$	$\mathcal{B}_2^-$	$\mathcal{B}_2^{cen}$	$\mathcal{B}_2^{cen-}$

References

[AS1] M. Aschbacher and Y. Segev, Uniqueness of sporadic groups, Groups, combinatorics and geometry (Durham, 1990), London Math. Soc. Lecture Note Ser. 165, Cambridge Univ. Press (1992), 1–11. MR1200245 (94e:20021)

[AS2] M. Aschbacher and Y. Segev, Extending morphisms of groups and graphs, Annals of Math. 135 (1992), 297–323. MR1154595 (93d:20034)

[Be] D. Benson, Representations and cohomology II: cohomology of groups and modules, Cambridge Univ. Press (1991). MR1156302 (93g:20099)

[BS] D. Benson and S. Smith, Classifying spaces of sporadic groups, Mathematical Surveys & Monographs 147, Amer. Math. Soc. (2008). MR2378355 (2009f:55017)

[Bc] S. Bouc, Homologie de certains ensembles ordonnés, Comptes Rendus Acad. Sci. Paris Série I 299 (1984), 49–52. MR756517 (85k:20150)

[Bu] F. Buekenhout, Diagrams for geometries and groups, J. Combin. Theory Ser. A 27 (1979), 121–151. MR542524 (83f:51003)

[Dw1] W. Dwyer, Homology decompositions for classifying spaces of finite groups, Topology 36 (1997), 783–804. MR1432421 (97m:55016)

[Dw2] W. Dwyer, Sharp homology decompositions for classifying spaces of finite groups, Proc. Symp. Pure Math. 63, Amer. Math. Soc. (1998), 197–220. MR1603159 (99b:55033)

[Gr] J. Grodal, Higher limits via subgroup complexes, Annals of Math. 155 (2002), 405–457. MR1906592 (2003g:55025)

[GS] J. Grodal and S. Smith, Propagating sharp homology decompositions, Advances in Math. 200 (2006), 525–538. MR2200855 (2007e:20110)

[Iv1] A. Ivanov, The geometry of sporadic groups, I: Petersen and tilde geometries, Encyclopedia of Mathematics and Its Applications 76, Cambridge Univ. Press (1999). MR1705272 (2000i:20028)

[Q] D. Quillen, Homotopy properties of the poset of nontrivial p-subgroups of a finite group, Adv. in Math. 28 (1978), 101–128. MR493916 (80k:20049)

[RS] M. Ronan and S. Smith, 2-local geometries for some sporadic groups, Santa Cruz Conference on Finite Groups, Proc. Symp. Pure Math. 37, American Math. Soc. (1980), 283–289. MR604595 (82c:20033)

[Sw1] M. Sawabe, 2-radical subgroups of the Conway simple group Co_1, J. Algebra 211 (1999), 115–133. MR1656574 (99k:20038)

[Sw2] M. Sawabe, The centric p-radical complex and a related p-local geometry, Math. Proc. Camb. Phil. Soc. 133 (2002), 383–398. MR1919712 (2003m:20014)

[So] L. Solomon, The Steinberg character of a finite group with a BN-pair, Theory of Finite Groups, Benjamin (1969), 213–221. MR0246951 (40:220)

[Ti] J. Tits, Les groupes de Lie exceptionnels et leur interpretation geometrique, Bull. Soc. Math. Belg. 8 (1956), 48–81. MR0087889 (19:430d)

[Wb] P. Webb, A split exact sequence of Mackey functors, Comment. Math. Helv. 66 (1991), 34–69. MR1090164 (92c:20095)

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