BOOK REVIEWS

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Outer billiards on kites, by Richard Evan Schwartz, Annals of Mathematics Studies, 171, Princeton University Press, Princeton, New Jersey, 2009, xiv+306 pp., ISBN 978-0-691-14249-4

Mathematical billiards is a popular object of study: many hundreds of papers devoted to billiards have appeared in mathematical and physical journals over the years. A lesser-known dynamical system, *outer billiards*—also known as dual billiards and antibilliards—has attracted a substantial attention in the last two decades. The book under review is the first research monograph devoted to the subject; it provides an in-depth analysis of outer billiards on a class of quadrilaterals called kites. The complexity and beauty of the emerging picture is stunning!

The outer billiard table is a convex domain in the plane bounded by a closed curve γ, oriented clockwise. Unlike conventional, or inner, billiards, the game is played outside of the table. Pick a point x in the exterior of γ, draw the tangent line to γ that agrees with the orientation, and reflect x in the tangency point to obtain a new point, y. The map $T : x \mapsto y$ is the outer billiard map; see Figure 1.

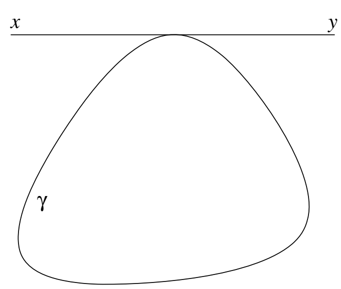

FIGURE 1. The outer billiard transformation

If the tangency point is not unique, that is, the tangent line has a whole interval in common with the table, the outer billiard map is not defined. This shortcoming is akin to the fact that one cannot consistently define the billiard reflection in a corner of a billiard table. Fortunately, the set of points where T is not defined has zero measure, so one still has plenty of room to play the game. In particular, this concerns polygonal outer billiards, the subject of the book [15] of R. Schwartz.

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An immediate consequence of the definition is that the outer billiard system commutes with affine transformations of the plane. For example, from the point of view of outer billiards, there is no difference between a circle and an ellipse. This greater symmetry group, compared to the group of similarities in the case of inner billiards, sometimes simplifies the study of outer billiards. For example, it is an outstanding open problem whether every polygonal billiard has a periodic orbit (the world record again belongs to R. Schwartz who proved that every obtuse triangle with its obtuse angle not greater than 100° has a periodic billiard trajectory [12, 14]); for polygonal outer billiards, this is a relatively simple, albeit recent, theorem [21]. Another example: it is known that the complexity (in the sense of symbolic dynamics) of polygonal billiards is subexponential [1], but it is unknown whether it is polynomial; in contrast, the polygonal outer billiard complexity is known to be polynomial [6].

Note that the definition of outer billiards makes sense in other classical geometries: the spherical and the hyperbolic ones. In the spherical world, the system is dual to the conventional, inner, billiards with respect to the spherical duality that interchanges great circles with their poles (this explains the name "dual billiards"). Outer billiards in the hyperbolic plane remain largely unexplored. The reader may wonder whether outer billiards can be also defined in a multidimensionall setting; the answer is yes, in even-dimensional spaces, and the outer billiard map is symplectic. For this and other aspects of outer billiards, we refer to the survey article [2] and the respective chapters of the books [19, 20].

Outer billiards were introduced by B. Neumann [11]. In particular, Neumann asked whether the orbits of the outer billiard map can escape to infinity. The subject owes its popularity to J. Moser who realized that the outer billiard map is area preserving and hence can be studied by the methods of Kolmogorov-Arnold-Moser (KAM) theory. In his classic book [9], Moser outlined a proof of the theorem that gives a partial answer to Neumann's question: if the outer billiard curve γ is strictly convex and sufficiently smooth, then the outer billiard map has invariant curves arbitrarily far away from the table, and therefore all orbits of the map stay bounded. Moser returned to outer billiards in his expository article [10] where he treated this system as a crude model for planetary motion. Moser reiterated the problem: can polygonal outer billiard orbits escape to infinity?

The first step toward solution was made in three papers by five authors [22, 7, 5]. To explain the result, we need to say a few words about the behavior of the second iteration of the outer billiard map far away from the table. Imagine a bird's eye view of outer billiards: the table is almost a point (the origin), the map T is almost a reflection in the origin. However, the second iteration of the map, T^2, takes one almost back to the starting point, and the iteration of the map T^2 appears a continuous motion along a centrally symmetric closed curve. To continue with the astronomical analogy, this continuous motion satisfies Kepler's second law: the area swept by the position vector of a point depends linearly on time.

This asymptotic motion is especially easy to understand if the outer billiard curve γ is an n-gon. The outer billiard map is a reflection in a vertex, and its second iteration is a parallel translation through a vector that equals twice a diagonal of γ. The discontinuities of T^2 are $2n$ rays: the counterclockwise extensions of the sides of γ and the reflections of these rays in the opposite vertices of γ (a vertex opposite to a side is the one farthest from it). The lines containing these $2n$ rays form n strips whose intersection contains γ; see Figure 2. In this figure, an origin

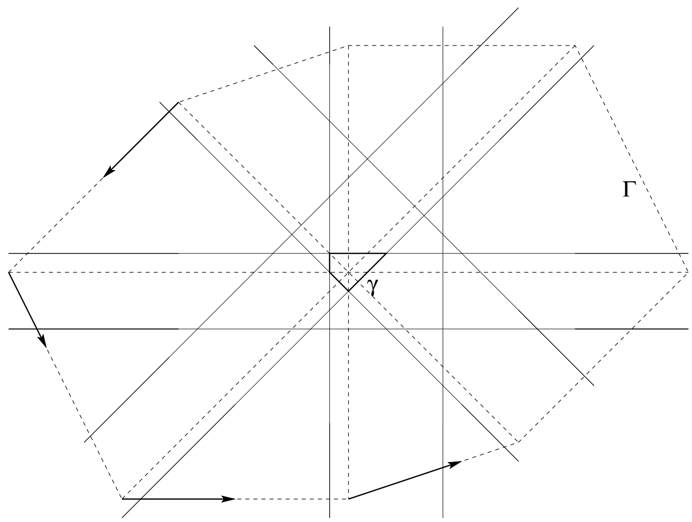

FIGURE 2. Polygonal outer billiard trajectory "at infinity"

is chosen inside γ, and the asymptotic trajectory at infinity is an origin symmetric octagon Γ with the vectors of translation along the sides indicated. This octagon is defined up to a dilation.

In general, one assigns the "time" to each side of the trajectory at infinity Γ, the ratio of the length of this side to the magnitude of the respective translation vector. One obtains a collection of times $(t_1, \dots, t_n)$, defined up to a common factor. The polygon is called *quasi-rational* if all these numbers are rational multiples of each other. In particular, lattice polygons are quasi-rational, and so are regular polygons. The theorem, proved in [22, 7, 5] in three different ways, asserts that the orbits of quasi-rational polygonal outer billiards stay bounded. As a consequence, if the outer billiard table is a lattice polygon then all orbits are periodic (being discrete and bounded). An elementary proof of this seemingly simple result is not known.

Let us emphasize that the outer billiard dynamics on quasi-rational polygons is far from trivial and, so far, is very poorly understood, even in the case of affine-regular polygons. There are two exceptions: the regular pentagon and octagon. The former was studied by the author in [18], and the latter, very recently and in much more depth, by R. Schwartz [17]. See Figure 3 for infinite orbits in these systems shown as a black "web", the white polygons are periodic domains.

Let us mention other results on outer billiard orbits escaping to infinity. It was independently proved in [4] and [8] that the outer billiard orbits for trapezoids always stay bounded. It was observed in computer experiments in the early 1990s that escaping orbits exist for the outer billiard on a semicircle: in fact, there is an open set that systematically spirals away to infinity. This was proved only recently by Dolgopyat and Fayad [3], and their techniques also apply to the regions obtained from a disk by nearly cutting it in half with a straight line. As we shall see, the mode of escape to infinity for kites is much more complicated.

Before describing the content of Schwartz's book, a few words about his approach to the problem. The work is a fine example of experimental mathematics: most of the phenomena were discovered through computer experimentation with the computer program *Billiard King* created by Schwartz for this purpose. This heavily documented program is available at `www.math.brown.edu/~res/BilliardKing/index.html`; one can download it or use it online. Using this program is highly recommended in conjunction with reading the book. Some of the proofs in the book are also computer assisted, but everything can be verified by hand. To quote from [15],

> One could do these calculations by hand in the same way that one could count all the coins filling up a bathtub. One could do it, but it is better left to a machine.

I would like to add that such a monumental task as counting coins in a bathtub by hand is prone to produce errors. The fact that the results in the book are directly observable via *Billiard King* is kind of a certificate of the validity of the proofs. Schwartz is a rare master of computer-inspired research; an interested reader is invited to visit his web site and to explore about 60 Java applets available therein.

A *kite* is a convex quadrilateral, one of whose diagonals is its axis of symmetry (rhombi do not count as kites). Any kite is affinely equivalent to the one with the vertices at $(-1,0),(0,1),(0,-1),(A,0)$ with $A \in (0,1)$. The number A is a parameter; a kite is irrational if so is A. The best known of all kites is the Penrose kite, with $A = \sqrt{5}-2$, which appears in the famous Penrose kite-and-dart quasi-periodic tiling. Schwartz studies outer billiards on the Penrose kite in [13], a predecessor of the book under review.

Although outer billiards on kites are 2-dimensional maps, the book concerns the restriction on the invariant countable union of horizontal lines $\mathbb{R} \times \mathbb{Z}_{odd}$ where $\mathbb{Z}_{odd}$ is the set of odd integers. The restriction of the map on this invariant set is an exchange of an infinite collection of intervals. An orbit is called *special* if it belongs to this invariant set. An orbit is *forward* (*backward*) *erratic* if the forward (backward) orbit is unbounded but also returns to every neighborhood of a vertex of the kite. An orbit is *erratic* if it is both forward and backward erratic.

Theorem 0.1 (Erratic Orbits)**.** *For any irrational kite,*

1. *there are uncountably many erratic special orbits;*
2. *every special orbit is either periodic or unbounded in both directions;*

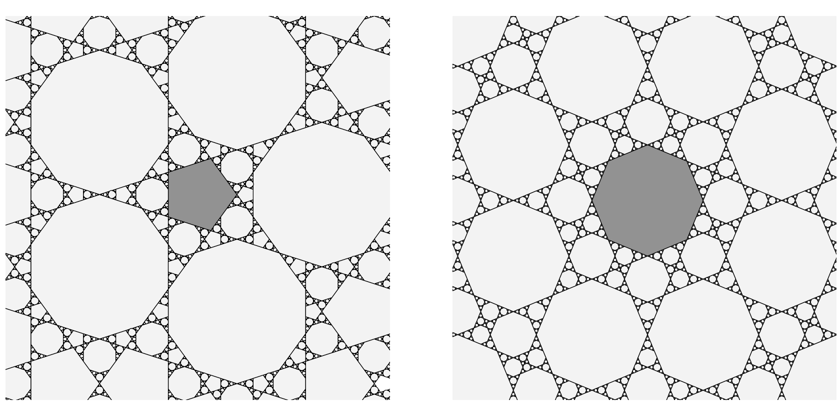

FIGURE 3. Outer billiards on a regular pentagon and an octagon

3. *the set of periodic special orbits is open and dense in* $\mathbb{R} \times \mathbb{Z}_{odd}$.

As a consequence, outer billiards on a kite has an unbounded orbit if and only if the kite is irrational.

The main result of the book is a theorem that Schwartz calls the Comet Theorem (once again, using a celestial analogy). This theorem is too technical to formulate here; it describes a renormalization scheme closely related to continued fractions. We formulate some consequences of the Comet Theorem that considerably sharpen the Erratic Orbits Theorem.

Let $I = [0, 2] \times \{-1\}$; this is a subinterval of the invariant set. Let U_A denote the set of unbounded special orbits relative to the kite parameter A. For a Cantor set C on a line L, let $C^\#$ be the set obtained from C by deleting the endpoints of the components of $L - C$; this is called a *trimmed Cantor set*.

Theorem 0.2. *For any irrational* $A \in (0, 1)$,

1. U_A *is minimal*: *every orbit in* U_A *is dense in* U_A *and all but at most two orbits in* U_A *are both forward and backward dense in* U_A*;*

2. U_A *is locally homogeneous*: *every two points in* U_A *have arbitrarily small isometric neighborhoods;*

3. $U_A \cap I = C_A^\#$ *for some Cantor set* C_A.

The Comet Theorem describes the Cantor set C_A explicitly.

Let ρ_A be the first return map to $U_a \cap I$. To describe the dynamics of ρ_A, we need a couple of definitions.

For $j = 1, 2$, let $f_j : X_j \to X_j$ be a map such that f_j and f_j^{-1} are defined everywhere, except possibly a finite set. Then f_1 and f_2 are called *essentially conjugate* if there exist countable sets $C_j \subset X_j$, each contained in a finite union of orbits, and a homeomorphism $h : X_1 - C_1 \to X_2 - C_2$ that conjugates f_1 to f_2.

An *odometer* is the map $x \mapsto x + 1$ on the inverse limit of the system

$$\cdots \mathbb{Z}/d_3 \to \mathbb{Z}/d_2 \to \mathbb{Z}/d_1, \quad \text{where} \quad d_k | d_{k+1}.$$

The universal odometer corresponds to $d_k = k!$.

Theorem 0.3.

1. *For any irrational* $A \in (0, 1)$*, the map* ρ_A *is defined on all but at most one point and is essentially conjugate to an odometer* $\mathcal{Z}_A$.
2. *Any odometer is essentially conjugate to* ρ_A *for uncountably many different choices of* A.
3. ρ_A *is essentially conjugate to the universal odometer for almost all* A.

The Comet Theorem describes the odometer $\mathcal{Z}_A$ explicitly.

The next result describes a surprising connection between outer billiards on kites and the modular group. Let Γ be the $(2, \infty, \infty)$-triangle group, that is, the group of isometries of the hyperbolic plane (in the upper half-plane model) generated by reflections in the sides of the triangle with vertices $(0, 1, i)$. In the next theorem, the kite parameter is interpreted as belonging to the ideal boundary of $\mathbb{H}^2$. Let $S = [0, 1] - \mathbb{Q}$, and let $u(A)$ be the Hausdorff dimension of U_A.

Theorem 0.4.

1. *For all* $A \in S$*, the set* U_A *has length 0: almost all points in* $\mathbb{R} \times \mathbb{Z}_{odd}$ *have periodic points relative to* A.

2. *if A and A' are in the same Γ-orbit then U_A and $U_{A'}$ are locally similar, and hence $u(A) = u(A')$.*
3. *if $A \in S$ is quadratic irrational then every point of U_A lies in an interval that intersects U_A is a self-similar trimmed Cantor set.*
4. *the function u is almost everywhere equal to some constant and yet maps every open subset of S onto $[0, 1]$.*

Let us say a few words about the central construction in the book, the *arithmetic graph*. This is a 2-dimensional representation of the first return map to the set $\Xi = \mathbb{R}_+ \times \{-1, 1\}$. Denote this first return map by Ψ; this is also an infinite interval exchange map. For ease of explanation, assume that A is a rational number p/q. Define the map $F : \mathbb{Z}^2 \to 2\mathbb{Z}[A] \times \{-1, 1\}$ by the formula

$$F(m, n) = (2Am + 2n + 1/q, (-1)^{p+q+1}).$$

The graph $\hat{\Gamma}(p/q)$ is formed by joining the points (m_1, n_1) and (m_2, n_2) if they are sufficiently close to each other and $F(m_1, n_1) = \Psi^{\pm 1} F(m_2, n_2)$. The main interest is in the component that contains $(0, 0)$ which is denoted by $\Gamma(p/q)$. When both p and q are odd, $\Gamma(p/q)$ is an infinite periodic polygonal arc, invariant under translation by the vector $(q, -p)$. Here are some structural results about the arithmetic graph, formulated informally:

1. **The Embedding Theorem**: $\hat{\Gamma}(p/q)$ is a disjoint union of embedded polygons and infinite embedded polygonal arcs. Every edge has length at most $\sqrt{2}$.

2. **The Hexagrid Theorem**: the structure of $\hat{\Gamma}(p/q)$ is controlled by six infinite families of parallel lines.

3. **The Copy Theorem**: if A_1 and A_2 are two rationals that are close in the sense of Diophantine approximation, then the corresponding arithmetic graphs Γ_1 and Γ_2 have substantial agreement.

See Figure 4.

The analysis of the arithmetic graph goes by way of the following result.

Theorem 0.5 (Master Picture). *For every A,*

1. *there is a locally affine map μ from Ξ to the union $\hat{\Xi}$ of two 3-dimensional tori;*
2. *there is a polyhedron exchange map $\hat{\Psi} : \hat{\Xi} \to \hat{\Xi}$ defined relative to a partition of $\hat{\Xi}$ into 28 polyhedra;*
3. *the map μ is a semiconjugacy between Ψ and $\hat{\Psi}$.*

There is a unifying 4-dimensional master picture, a union of two convex lattice polytopes partitioned into 28 smaller convex lattice polytopes; for each parameter A, one obtains the 3-dimensional picture as a section. Schwartz says that his study

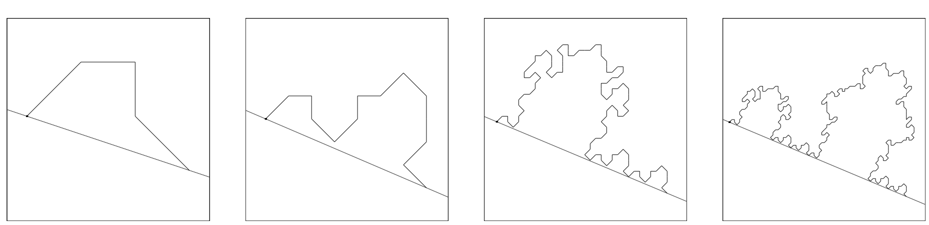

FIGURE 4. The graphs $\Gamma(1/3), \Gamma(3/7), \Gamma(13/31)$, and $\Gamma(29/69)$

of the Master Picture Theorem is just starting, and that a version of this result should hold in much more generality. See [16] for a generalization to other polygons of one of the results from the book, the Pinwheel Lemma, that concern strip maps such as the one depicted in Figure 2.

Let me conclude with a general impression of the research presented in the book. Its main feature is a spirit of exploration. One enters empty-handed, with only a vague idea of what one might encounter, and one exits, after a long expedition, with a fulfilling detailed picture of a highly structured and beautiful world, full of surprises and the previously unseen.

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