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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
SIMON BRENDLE AND RICHARD SCHOEN

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Hamilton, Richard S.

The formation of singularities in the Ricci flow. (English)

Surveys in differential geometry, Vol. II (*Cambridge, MA,* 1993), 7–136, *Int. Press, Cambridge, MA*, 1995.

In this paper the author surveys some of the basic geometrical properties of the Ricci flow with a view to considering what kind of singularities might form. The Ricci flow was introduced by the author in his celebrated work on deformation of metrics the by Ricci flow on compact 3-manifolds with positive Ricci curvature [J. Differential Geom. **17** (1982), no. 2, 255–306; MR0664497 (84a:53050)]. It has been found to be a useful tool in the study of Riemannian geometry, both for compact and for open manifolds. The idea is to study the evolution of a Riemannian metric along its Ricci curvature. Many interesting results have been obtained using the Ricci flow, and it is most desirable for the author to have written a survey in the field where he initiated and continues to make contributions. This article is a beautiful example of how to use techniques in differential equations and Riemannian geometry to study manifolds.

The paper starts with some examples of how singularities might be formed under the Ricci flow. Short-time existence, derivative estimates and long-time existence are reviewed. Then the author discusses the convergence of the Ricci flow on dimension two, three and four, as well as on Kähler manifolds. It is known that the Ricci flow is invariant under the full diffeomorphism group and any isometries in the initial metric will persist as isometries in each subsequent Riemannian metric. The author illustrates the idea by considering a Riemannian metric on a 3-manifold where the torus group $T^2 = S^1 \times S^1$ acts freely as a subgroup of the isometry group. It is shown that the Ricci flow of such a metric exists for all time and converges to a flat metric. The author discusses an estimate on the derivatives of curvature from local conditions and the Harnack inequality for the Ricci flow. He gives a discussion on the limits of solutions to the Ricci flow. To do so, a lower bound on the injectivity radius at a point is obtained in terms of a local bound on the curvature. The author also gives a control on the distance between two fixed points along the Ricci flow.

Then the author shows that some asymptotic properties of complete Riemannian manifolds are preserved by the Ricci flow. Define the aperture α of a complete non-compact Riemannian manifold by $\alpha = \limsup_{s\to\infty} \operatorname{diam} S_s/2s$, where $\operatorname{diam} S_s$ is the diameter of the sphere with radius s. It is shown that for a complete solution to the Ricci flow with bounded curvature and weakly positive Ricci curvature, the aperture α is a constant. If the Ricci flow on a complete non-compact Riemannian manifold has bounded curvature and the Riemannian curvature is approaching zero at infinity at time zero, then this remains true along the Ricci flow. Suppose we have a complete solution to the Ricci flow on a complete Riemannian n-manifold with bounded curvature and weakly positive Ricci curvature, where the Riemannian curvature is approaching zero at infinity; then the asymptotic volume ratio ν defined by $\nu = \lim_{s\to\infty} \operatorname{Vol} B_s/s^n$ is constant along the Ricci flow, where $\operatorname{Vol} B_s$ is the volume of the ball with radius s. Another asymptotic property that is preserved by the Ricci flow is the asymptotic scalar curvature ratio A defined by $A = \limsup_{s\to\infty} Rs^2$, where R is the scalar curvature. The author shows that for a complete solution to the Ricci flow with bounded curvature defined for $-\infty < t < T < \infty$, if the initial Riemannian metric either has positive curvature operator or is Kähler with weakly positive holomorphic bisectional curvature, then the asymptotic scalar curvature ratio A is constant.

The author investigates the influence of a bump of strictly positive curvature in a complete Riemannian manifold of weakly positive curvature. For any $\epsilon > 0$ we can find $\lambda < \infty$ such that for a complete Riemannian manifold with sectional curvatures $K \geq 0$, if P is a point such that $K \geq \epsilon/r^2$ everywhere in $B_{3r}(P)$, if $s \geq r$ and Q_1 and Q_2 lie outside $B_{\lambda s}(P)$, and if γ is a minimal geodesic from Q_1 to Q_2, then γ stays outside $B_s(P)$. This is used to prove the following result. Suppose that we have a solution to the Ricci flow on a compact m-manifold with weakly positive curvature operator for a maximal time interval $0 \leq t < T$; then we can find a sequence of dilations which converges to a complete solution of the Ricci flow with curvature bounded at each time on a time interval $-\infty < t < \Omega$ with scalar curvature R bounded by $R \leq \Omega/(\Omega - t)$ everywhere and $R = 1$ at some origin O at time $t = 0$, which again has weakly positive curvature operator. Moreover the limit splits as a quotient of a product $N \times \mathbf{R}^k$ with $m = n + k$ and $k \geq 0$, where the factor N is an n-manifold which either is compact or has finite asymptotic curvature ratio.

Then the author turns to dimension three and discusses the isoperimetric ratio bound and curvature pinching. He concludes with the following description on the formation of singularities of the Ricci flow. Suppose that we have a solution to the Ricci flow on a compact 3-manifold and suppose that the scalar curvature becomes unbounded in some finite time T. Then there exists a sequence of dilations of the solution converging to S^3 or $S^2 \times \mathbf{R}^1$ or $\Sigma^2 \times \mathbf{R}^1$, where Σ is the surface $(\mathbf{R}^2,\ dx^2 + dy^2/(1 + x^2 + y^2))$, or to a quotient of one of these manifolds by a finite group of isometries acting freely, except possibly for the case where the injectivity radius times the square root of the maximum curvature goes to zero.

The paper is well written and delightful to read. Many examples are worked out in detail to illustrate the idea. It gives a rather complete account and up-to-date references on current research on the Ricci flow.

From MathSciNet, September 2010

Man Chun Leung

MR1484888 (98k:53044) 53C21; 53C20

Cheeger, Jeff; Colding, Tobias H.

On the structure of spaces with Ricci curvature bounded below. I. (English)

J. Differential Geom. **46** (1997), *no.* 3, 406–480.

The paper under review is the first in a series of papers by Cheeger and Colding on the geometry and topology of spaces with a lower Ricci curvature bound. Typically, by a space with a lower Ricci curvature bound they mean the Gromov-Hausdorff limit of a sequence of manifolds with the same lower Ricci curvature bound. Several of the tools were developed by Cheeger and Colding in their earlier paper [Ann. of Math. (2) **144** (1996), no. 1, 189–237; MR1405949 (97h:53038)]. In addition to applying to singular limits, the results of these papers also have important applications to manifolds with lower Ricci curvature bounds. These papers, and many of these results, were announced in another paper by the authors [C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 3, 353–357; MR1320384 (96f:53047)].

Manifolds with lower Ricci curvature bounds have been extensively studied, and it would be impossible to do justice to all of the work in this area. We will, however, attempt to briefly survey some of the main tools for studying manifolds M^n with $\mathrm{Ric}_M \geq 0$. The first is the Bochner formula: if u is a harmonic function, i.e. $\Delta u = 0$, then

$$\Delta\,|\nabla u|^2 \geq 2|\nabla^2 u|^2 + 2\,|\nabla u|^2 \inf_{x\in M} \mathrm{Ric}_x \geq 0. \tag{1}$$

Second, the Laplacian comparison theorem states that $\Delta r \leq (n-1)/r$, where r is the distance function. This can be used to derive the relative volume comparison theorem: Given $x \in M^n$ with $\mathrm{Ric}_M \geq 0$, then

$$\frac{d}{ds}\left(\frac{\mathrm{Vol}\,(B_s(x)\subset M)}{\mathrm{Vol}\,(B_s\subset \mathbf{R}^n)}\right) \leq 0, \tag{2}$$

and, in particular, $\mathrm{Vol}\,(B_s(x)\subset M) \leq \mathrm{Vol}\,(B_s \subset \mathbf{R}^n)$ (Bishop's volume comparison theorem). If M admits a line then, by the splitting theorem of Cheeger and D. Gromoll [J. Differential Geometry **6** (1971/72), 119–128; MR0303460 (46 #2597)], the associated Busemann function is affine (and hence M splits isometrically). More recently, U. Abresch and Gromoll [J. Amer. Math. Soc. **3** (1990), no. 2, 355–374; MR1030656 (91a:53071)] used a similar analysis to study thin triangles and the excess. Another important classical result is the gradient estimate of S. Y. Cheng and S. T. Yau [Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354; MR0385749 (52 #6608)] which gives an a priori estimate for the gradient of a harmonic function.

M. Gromov's compactness theorem gives a useful framework for studying these manifolds [*Structures métriques pour les variétés riemanniennes*, Edited by J. Lafontaine and P. Pansu, CEDIC, Paris, 1981; MR0682063 (85e:53051)]. The Riemannian distance function dist allows us to view Riemannian manifolds as metric spaces (in fact, length spaces). Recall that if (X,d) is a metric space then the Hausdorff distance d_H between $A, B \subset X$ is less than ϵ if

$$A \subset \{x \in X|\ d(x,B) < \epsilon\} \text{ and } B \subset \{x\in X|\ d(x,A) < \epsilon\}.$$

The Gromov-Hausdorff distance d_{GH} between metric spaces X and Y generalizes this notion by defining $d_{\mathrm{GH}}(X,Y) = \inf_{(f,g,Z)} d_H(f(X), g(Y))$ where $f\colon X \to Z$ and $g\colon Y \to Z$ are isometric. Gromov's compactness theorem is the assertion that the set of metric spaces

$$\{(M^n, \mathrm{dist})|\ \mathrm{Ric}_M \geq -1,\ \mathrm{diam}_M \leq D\}$$

is precompact in the Gromov-Hausdorff topology. Similarly, if $p_j \in M_j$ is a sequence of manifolds with $\mathrm{Ric}_{M_j} \geq 0$ a subsequence, $B_1(p_{j'}) \subset M_{j'} \to B_1(p) \subset X$ in the pointed Gromov-Hausdorff topology where X is a length space. In the important special case when M_j is the rescaling $(M, r_j^{-2}\,g)$ of a fixed manifold (M, g) with $\mathrm{Ric}_M \geq 0$, such a limit X is called a tangent cone. If $r_j \to 0$ we get the tangent cone at a point, and when $r_j \to \infty$ we get the tangent cone at infinity. Since this metric space X is generally not a Riemannian manifold, this raises the question of what, if any, additional structure it has.

When the manifolds M_j have sectional curvature bounded from below then the triangle comparison theorems can be used to show that the limit X has a great deal of structure [Yu. D. Burago, M. L. Gromov and G. Ya. Perel′man, Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222; MR1185284 (93m:53035)]. In the case of Ricci curvature, these pointwise estimates fail and much less was known. The first *integral* estimates on distances and angles were developed by Colding [Invent. Math. **124** (1996), no. 1-3, 175–191; MR1369414 (96k:53067); Invent. Math. **124** (1996), no. 1-3, 193–214; MR1369415 (96k:53068); Ann. of Math. (2) **145** (1997), no. 3, 477–501; MR1454700 (98d:53050)]. This led to the solutions of well-known conjectures of Anderson-Cheeger, Gromov, and Fukaya-Yamaguchi.

The following example illustrates the various viewpoints. Let M^n be a closed manifold with $\mathrm{Ric}_M \geq 0$. By integrating (1), Bochner showed that $b_1(M^n) \leq n$ with equality for a flat torus. Applying (2) to the universal cover of M, J. Milnor showed that $\pi_1(M)$ has polynomial growth [J. Differential Geometry **2** (1968), 1–7; MR0232311 (38 #636)]. The splitting theorem of Cheeger and Gromoll implies that $\pi_1(M)$ is almost crystallographic. Suppose now that $\mathrm{diam}_M = 1$ and $\mathrm{Ric}_M \geq -\epsilon$. Gromov showed that there is some $\epsilon_1 > 0$ so that $\epsilon < \epsilon_1$ implies that $b_1(M^n) \leq n$. Settling a conjecture of Gromov, Colding showed that there is some $\epsilon_2 > 0$ such that $\epsilon < \epsilon_2$ and $b_1(M^n) = n$ implies that M is a torus.

In [op. cit., 1996], Cheeger and Colding proved quantitative generalizations of several rigidity theorems for Riemannian manifolds of positive (or nonnegative) Ricci curvature. Included in this were the volume cone implies metric cone theorem, the maximal diameter theorem, and the splitting theorem. It is perhaps useful to recall the first and third above.

From the volume comparison theorem, if $\mathrm{Ric}_M \geq 0$ and the ratio in (2) is constant for $s < r < t$ then $B_t \backslash B_r \subset M$ is a metric cone. That is, there is a closed manifold $(N^{n-1}, d\omega^2)$ such that $B_t \backslash B_r$ is isometric to $(N \times [s,t], dr^2 + r^2\,d\omega^2)$. Cheeger and Colding proved that if the ratio is almost constant then $B_t \backslash B_r \subset M$ is Gromov-Hausdorff close to a metric cone. As a consequence, they showed that if M^n has $\mathrm{Ric}_M \geq 0$ and Euclidean volume growth (namely, $\mathrm{Vol}(B_r) \geq cr^n$ for all $r > 0$ and some $c > 0$) then every tangent cone at infinity M_∞ is a metric cone of Hausdorff dimension n.

It follows from their almost splitting theorem that if $M_i^n \to Y$ in the pointed Gromov-Hausdorff distance where $\liminf \mathrm{Ric}_{M_i} \geq 0$ and Y contains a line then Y splits isometrically as $\mathbf{R} \times X$ for some length space X of Hausdorff dimension at most $n-1$. This is important for the structure theory of singular limits, but it also has implications for classical geometry. For instance, Cheeger and Colding used this to show that there exists some $\delta > 0$ such that if $\mathrm{diam}(M) \leq 1$ and $\mathrm{Ric}_M \geq -(n-1)\,\delta$ then $\pi_1(M^n)$ contains a nilpotent subgroup of finite index (answering a conjecture of Gromov).

In the present paper, the authors combine the tools developed in their first paper with techniques similar to those used in geometric measure theory to analyze the structure of spaces with Ricci curvature bounded from below. Suppose now that a length space (Y, d) is the pointed Gromov-Hausdorff limit $(M_i^n, p_i) \to (Y, p)$ where $\mathrm{Ric}_{M_i} \geq (n-1)\Lambda$ for all i. We recall next some of their definitions (which differ from those of the announcement). Define the weakly k-Euclidean set by $y \in \mathcal{W}_k$ if *some* tangent cone at y splits off a factor $\mathbf{R}^k$. If moreover *every* tangent cone at y is isometric to $\mathbf{R}^k$ then y is in the k-regular set $\mathcal{R}_k$. Let $\mathcal{R} = \bigcup_k \mathcal{R}_k$ denote the set of regular points. Define the k-degenerate set $\mathcal{D}_k$ by $\mathcal{D}_k = Y \backslash \mathcal{R}_{k+1}$, $\mathcal{S}_k$ to be $\mathcal{D}_k \backslash \mathcal{R}$, and the singular set $\mathcal{S}$ by $\mathcal{S} = \bigcup \mathcal{S}_k = Y \backslash \mathcal{R}$. The authors show first that the set of regular points $\mathcal{R} \subset Y$ is dense. Similar definitions are very important in regularity theory for harmonic maps and minimal surfaces (see, for instance, H. Federer [*Geometric measure theory*, Springer-Verlag New York Inc., New York, 1969; MR0257325 (41 #1976)], R. Schoen and K. Uhlenbeck [J. Differential Geom. **17** (1982), no. 2, 307–335; MR0664498 (84b:58037a); correction MR0710058 (84b:58037b)], or the more recent survey of L. M. Simon [in *Nonlinear partial differential equations in differential geometry (Park City, UT, 1992)*, 185–223, Amer. Math. Soc., Providence, RI, 1996; MR1369589 (96k:58060)]).

In the first four sections, the authors study the general case (which includes the possibility that $\lim \mathrm{Vol}(B_1(p_i)) = 0$, i.e., the sequence is collapsing). They construct renormalized limit measures ν on Y which they show satisfy the appropriate volume comparison and $\nu(\mathcal{S}) = 0$.

If in addition $\mathrm{Vol}(B_1(p_i)) \geq v > 0$ for all i (the sequence is noncollapsing) then they show that Y has Hausdorff dimension n and the renormalized limit measure is a multiple of n-dimensional Hausdorff measure ("volume convergence"). This implies that the relative volume comparison theorem holds for Y. Furthermore, they show that every tangent cone is a metric cone (note that they give examples showing that the tangent cones need not be unique). They also show that $\dim \mathcal{D}_k \leq k$.

In the noncollapsing case, they define the ϵ-regular set by $y \in \mathcal{R}_\epsilon$ if the Gromov-Hausdorff distance between $B_1(0) \subset \mathbf{R}^n$ and $B_1(y_\infty) \subset Y_y$ is less than ϵ for every tangent cone Y_y at y. In other words, $y \in \mathcal{R}_\epsilon$ if all sufficiently small balls at y are (d_{GH}) close to balls in $\mathbf{R}^n$, and hence $\mathcal{R}_\epsilon$ contains an open neighborhood of the regular set $\mathcal{R}$. They show that if $\epsilon < \epsilon(n)$ is sufficiently small then the interior of $\mathcal{R}_\epsilon$ is homeomorphic to a smooth manifold. In fact, the metric is bi-Hölder equivalent to a smooth metric. The idea is that if $y \in \mathcal{R}_\epsilon$ then since $B_1(y_\infty)$ is close to $B_1(0) \subset \mathbf{R}^n$ it must have almost maximal volume. The relative volume comparison then implies that all sufficiently small subballs have almost maximal volume. It then follows (see the earlier discussion) that these subballs are also Gromov-Hausdorff close (on their own scales) to balls in $\mathbf{R}^n$. In other words, we have a "Reifenberg" condition [cf. E. R. Reifenberg, Acta Math. **104** (1960), 1–92; MR0114145 (22 #4972)], which allows the authors to construct the bi-Hölder homeomorphism.

In this noncollapsing case, they also show that $\mathcal{S} \subset \mathcal{S}_{n-2}$ and hence the singular set has codimension at least two (which is optimal). In the collapsed case there are examples with codimension-one singular sets. The authors also obtain various stronger results under various additional assumptions on the sequence (two-sided Ricci curvature bounds, Einstein manifolds); this uses a result of M. T. Anderson [Invent. Math. **102** (1990), no. 2, 429–445; MR1074481 (92c:53024)]. A future paper of the authors with G. Tian will contain further results in this direction (including sharper results for Kähler manifolds); see their announcement [C. R. Acad. Sci. Paris Sér. I Math. **324** (1997), no. 6, 645–649; MR1447035 (98g:53078)].

From MathSciNet, September 2010

William P. Minicozzi II

MR1815410 (2003a:53043) 53C21; 49Q15, 53C20, 53C23

Cheeger, Jeff; Colding, Tobias H.

On the structure of spaces with Ricci curvature bounded below. II.

Journal of Differential Geom. **54** (2000), *no.* 1, 13–35.

Often, simple examples can illuminate an abstract concept and build up intuition, serving as guides for further endeavor. Rauch and Toponogov introduced the idea of comparing the geometry of a Riemannian manifold with familiar models, such as spheres, Euclidean and hyperbolic spaces. People immediately recognized that this is a way to understand the elusive global properties of Riemannian manifolds (see the introductory book by J. Cheeger and D. G. Ebin [*Comparison theorems in Riemannian geometry*, North-Holland, Amsterdam, 1975; MR0458335 (56 #16538)]). The early developments were crystallized into the wonderful concept of Gromov-Hausdorff distance, which links two Riemannian manifolds M_1 and M_2 by an intermediate metric space Z through isometries $f\colon M_1 \to Z$ and $g\colon M_2 \to Z$, measuring how close they can get in this way.

Gromov also contributed a vital precompactness result. It says that a sequence of closed Riemannian n-manifolds $\{M_i^n\}$ with diameters bounded uniformly from above and Ricci curvature bounded uniformly from below has a subsequence converging to a limit space X. In case X is smooth, it provides a model to compare those M_i^n that are nearby. Thus, this apparently simple idea enables one to extend local geometric data into global information, shedding light on the finiteness of closed manifolds with bounded geometry, together with pinching and rigidity phenomena. A typical application can be found in [M. T. Anderson and J. Cheeger, Geom. Funct. Anal. **1** (1991), no. 3, 231–252; MR1118730 (92h:53052)], where the Gromov-Hausdorff convergence is used to generalize Cheeger's finiteness theorem.

In this series of articles, the authors address the primary concern of the structure of the limit space Y of a sequence $\{M_i^n\}$ of complete, connected Riemannian n-manifolds with $\mathrm{Ric}_{M_i^n} \geq -(n-1)$. The fine review written by W. P. Minicozzi II of the first paper [J. Differential Geom. **46** (1997), no. 3, 406–480; MR1484888 (98k:53044)] explains in depth the background of the study and the main results on the singular set of Y. Here two things are at work. Firstly, the assumption on Gromov-Hausdorff convergence provides information on the metric, but very little on measure. It is coupled with a lower bound on the Ricci curvature, which provides information on suitably chosen local coordinate charts.

These notions are made vivid by the main result of the paper, which generalizes the volume convergence theorem to the case of collapsed limits (that is, $\inf \mathrm{Vol}(B_1(m_i)) = 0$, where $B_1(m_i) \subset M_i^n$). Recall that in the case in which the limit space Y is a Riemannian n-manifold, the second author has shown that $\mathrm{Vol}(M_i^n) \to \mathrm{Vol}(Y)$ [Ann. of Math. (2) **145** (1997), no. 3, 477–501; MR1454700 (98d:53050)]. This was previously conjectured by Anderson and the first author.

In general, when the manifolds collapse (as in the case when Y has dimension smaller than n), the renormalized limit measure, which is defined as the limit of a subsequence of $\mathrm{Vol}_j(\cdot)/\mathrm{Vol}_j(B_1(m_j))$, $B_1(m_j) \subset M_j^n$, may not be unique. Thus the situation is far more complicated, as one is in search of a suitable measure which behaves stably under the Gromov-Hausdorff convergence.

A suitable notion is found to be the k-dimensional spherical Hausdorff content, $\mathcal{H}^k_\infty$. In the spirit of the Gromov-Hausdorff distance, it is defined through a process of atomization and minimization: For subsets U of a metric space Z,

$$\mathcal{H}^k_\infty(U) = V_k \inf_{\mathcal{B}} \sum_i r_i^k,$$

where V_k is the volume of the unit ball in $\mathbf{R}^k$ and r_i denotes the radius of the ball (in Z) in the covering $\mathcal{B} = \{B_{r_i}(q_i)\}$ of U. The authors show that if the sequence $\{M_i^n\}$ converges in the pointed Gromov-Hausdorff distance to M^k, where M^k is a manifold, then $\mathcal{H}^k_\infty(M_i^n) \to \mathcal{H}^k_\infty(M^k)$. The key is to construct a Lipschitz map $\Phi_i\colon B_1(m_i) \to B_1^k(0)$, where $B_1(m_i)$ converges in the pointed Gromov-Hausdorff distance to $B_1^k(0) \subset \mathbf{R}^k$ as $i \to \infty$, and to show that $\Phi_i(B_1(m_i))$ has almost full measure in $B_1^k(0)$ for i sufficiently large. Technically speaking, this involves a type of reverse Poincaré inequality and a weighted volume function.

Concerning whether the isometry group of Y is a Lie group: this is found to be true in the noncollapsed case, but remains conjectural in the collapsed case. Lastly, the authors show (with the help of an additional condition) that in case Y contains a 1-dimensional piece, then it is isometric to a 1-dimensional manifold with possibly nonempty boundary.

{For Part III see [J. Cheeger and T. H. Colding, J. Differential Geom. **54** (2000), no. 1, 37–74; MR1815411 (2003a:53044)].}

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Man Chun Leung

MR1815411 (2003a:53044) 53C21; 49Q15, 53C20, 53C23

Cheeger, Jeff; Colding, Tobias H.

On the structure of spaces with Ricci curvature bounded below. III.

Journal of Differential Geom. **54** (2000), *no.* 1, 37–74.

This third paper in the series concerns the spectral information of a space Y which is the pointed Gromov-Hausdorff limit of a sequence $\{M_i^n\}$ of complete, connected Riemannian n-manifolds with Ricci curvature uniformly bounded from below. A description on the singular set of Y is presented in the authors' first article [J. Differential Geom. **46** (1997), no. 3, 406–480; MR1484888 (98k:53044)]. In order to capture a sense of smoothness and to introduce the Laplacian on Y, they explore essential notions in geometric measure theory as developed in H. Federer's celebrated titanic book [*Geometric measure theory*, Springer-Verlag New York Inc., New York, 1969; MR0257325 (41 #1976)].

A set E in $\mathbf{R}^n$ is said to be m-rectifiable if there is a sequence of C^1 maps, $f_i\colon \mathbf{R}^m \to \mathbf{R}^n$, such that the m-Hausdorff measure of $E \setminus (\bigcup f_i(\mathbf{R}^m))$ is equal to zero. For a general metric space with a Radon measure μ, the concept of μ-rectifiability revolves around two insights. Firstly, after removing a set of measure zero, the remaining part of the space is a union of sets which are bi-Lipschitz to subsets in Euclidean spaces, and secondly, when the measure is restricted to each of the sets, it is absolutely continuous with respect to the relevant Hausdorff measure.

Recall that renormalized limit measures ν on Y arise as limits of subsequences of normalized Riemannian measures $\mathrm{Vol}_j(\cdot)/\mathrm{Vol}_j(B_1(m_j))$, where $m_j \in M_j^n$. In the case when $\mathrm{Vol}_j(B_1(m_j))$ is uniformly bounded away from zero (the noncollapsed case), the measure ν is unique and coincides with the normalized Hausdorff measure. In the collapsed case, the renormalized limit measure ν still exists, but may not be unique.

In order to control ν, the key here is to construct locally defined bi-Lipschitz maps from Y to Euclidean spaces, with a good grasp on the Lipschitz constants. It is shown that, with respect to any fixed such measure ν, Y is a finite union of countably ν-rectifiable spaces, and ν is absolutely continuous with respect to the relevant Hausdorff measure. It follows that the regular part of Y is a finite union of spaces which, although they are not necessarily subsets of Euclidean space, have the properties of countably rectifiable varifolds, whose dimensions might not all be the same. The conception of the work is of high originality and blends beautifully ideas in geometric measure theory and Riemannian geometry.

The authors show that there is a unique self-adjoint operator Δ (the Laplacian) on Y such that

$$\int |df|^2\, d\nu = \langle \Delta^{\frac{1}{2}} f,\ \Delta^{\frac{1}{2}} f\rangle$$

for Lipschitz functions. The continuity of the eigenvalues and eigenfunctions under the pointed Gromov-Hausdorff limit is demonstrated. By using the almost differentiability of Lipschitz functions, they also introduce differential forms on Y, locally expressed as $f_0\, df_1 \wedge \cdots \wedge df_i$, where f_i are Lipschitz functions. But the above continuity on functions cannot be extended to 2-forms, as there exists a (noncollapsing) sequence of compact manifolds of positive Ricci curvature with unbounded second Betti numbers, converging in the Gromov-Hausdorff distance to a compact space (cf. the recent works of G. Ya. Perel′man [in *Comparison geometry (Berkeley, CA, 1993–94)*, 157–163, Cambridge Univ. Press, Cambridge, 1997; MR1452872 (98h:53062)] and X. Menguy [Geom. Funct. Anal. **10** (2000), no. 3, 600–627; MR1779615 (2001g:53074)]).

{For Part II see [J. Cheeger and T. H. Colding, J. Differential Geom. **54** (2000), no. 1, 13–35; MR1815410 (2003a:53043)].}

From MathSciNet, September 2010

Man Chun Leung

MR2460872 (2010h:53098) 53C44; 57M40

Kleiner, Bruce; Lott, John

Notes on Perelman's papers.

Geometry & Topology **12** (2008), *no.* 5, 2587–2855.

Between November 2002 and July 2003, Grigori Perelman posted on the ArXiv three major papers ["The entropy formula for the Ricci flow and its geometric applications", preprint, `arxiv.org/abs/math/0211159`; "Ricci flow with surgery on three-manifolds", preprint, `arxiv.org/abs/math/0303109`; "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds", preprint, `arxiv.org/abs/math/0307245`]. They contained a sketchy (but concise) proof of the Poincaré conjecture and Thurston's geometrization conjecture. The approach uses the Ricci flow, introduced by Richard Hamilton in the 1980's. The texts, still available on the web, are revolutionary in many ways. Kleiner and Lott started to work on these articles in order to fill in the details and check the various statements. Preliminary notes were soon posted and made available to everyone interested in understanding this major breakthrough; they were also regularly updated. The present article is the final version crowning their work. It contains a very complete description of the first two papers posted by Perelman. The proof of Thurston's geometrization is given. It is certainly an unavoidable reference.

The authors have made the choice of following as closely as possible Perelman's approach. Each chapter, section or subsection refers to a part of Perelman's articles. They start with a beautiful and precise overview of the Ricci flow approach to the geometrization. Although the proof of the Poincaré conjecture is not directly addressed in these notes, the authors include a rough outline of it. Let me recall that there are two ways to derive the Poincaré conjecture from the Ricci flow. Either one proves the geometrization conjecture and Poincaré's question is answered as a by-product, or one shows that if one starts with a simply-connected manifold the Ricci flow stops in finite time (we say that it becomes extinct). This last approach is a short-cut since one does not have to go through the painful long time study of the Ricci flow. The finite time extinction is the third paper posted by Perelman [op. cit., "Finite extinction time ..."]. The argument is detailed in the book by J. W. Morgan and G. Tian [*Ricci flow and the Poincaré conjecture*, Amer. Math. Soc., Providence, RI, 2007; MR2334563 (2008d:57020)] which focuses on the proof of the Poincaré conjecture. A somewhat different solution for showing the finite time extinction was given by T. H. Colding and W. P. Minicozzi, II [J. Amer. Math. Soc. **18** (2005), no. 3, 561–569; MR2138137 (2006c:53068)]. The overview of the geometrization given in that text is sufficient for a reader who would only want to get the general picture. It describes, for example, the structure of singularities obtained by the blow-up technique relying on the compactness theorem and the non-collapsing property. Then follows a more precise summary of the first paper. After these very useful surveys the authors proceed to the description of each section of [G. Perelman, op. cit., "The entropy formula ..."]. The computations are given with full details, sometimes with alternative proofs, and some very nice examples enlightening the basic notions introduced by Perelman.

The description of the second paper, done equally carefully, is again preceded by a useful overview. The reader should be aware that this second part is technically difficult since it contains the construction of the Ricci flow with surgery and the study of its behaviour during long time evolution. After some practice one gets used to the proofs (mostly by contradiction) and the treatment given by the authors appears to be very clear. This should be read while looking also at the original papers by Perelman; the picture then clarifies progressively. In the last part the geometrization is proven. When the Ricci flow with surgery evolves, the manifold splits into a so-called thick part, that is, a non-collapsing part, which ought to become hyperbolic (with finite volume) after rescaling, and a thin part. The thin part is a graph manifold (with boundary). One important issue is that the hyperbolic pieces, if noncompact, are bounded by incompressible tori, that is tori whose fundamental group injects in the ambient manifold. Several proofs are given. One relies on an idea of Hamilton using minimal surfaces. Two others use the monotonicity of some Riemannian invariants: the first eigenvalue of a Schrödinger operator or the normalised minimum of the scalar curvature. The proof that the thin part is a graph manifold relies on a work by T. Shioya and T. Yamaguchi [Math. Ann. **333** (2005), no. 1, 131–155; MR2169831 (2006j:53050)]. Although this last paper concerns closed manifolds the method seems to go through in the case of manifolds with boundary. More precise treatments were proposed in [B. Kleiner and J. Lott, "Locally collapsed 3-manifolds", to appear; J. Morgan and G. Tian, "Completion of the proof of the geometrization conjecture", preprint, `arxiv.org/abs/0809.4040`]. A completely different approach can be read in the article [L. Bessières et al., Invent. Math. **179** (2010), no. 2, 435–460; MR2570121] and the book [L. Bessières et al., *Geometrisation of* 3*-manifolds*, European Mathematical Society (EMS), to appear]. Another article detailing the proof of the geometrization conjecture has been published by H.-D. Cao and X. P. Zhu [Asian J. Math. **10** (2006), no. 2, 165–492; MR2233789 (2008d:53090); erratum, Asian J. Math. **10** (2006), no. 4, 663; MR2282358 (2008d:53091)].

It is particularly interesting to use all these references, each of them adding a stone to the building.

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