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Fine structure of hyperbolic diffeomorphisms, by A. A. Pinto, D. Rand, and F. Ferreira, Springer Monographs in Mathematics, Springer-Verlag, Berlin, Heidelberg, 2009, xvi+354 pp., ISBN 978-3-540-87524-6, hardcover, US$129.00

The main theme of the book *Fine Structures of Hyperbolic Diffeomorphisms*, by Pinto, Rand and Ferreira, is the rigidity and flexibility of two-dimensional diffeomorphisms on hyperbolic basic sets and properties of invariant measures that are related to the geometry of these invariant sets.

In his remarkable article [23], Smale sets the foundations of the modern theory of dynamical systems. He defines the fundamental notion of hyperbolicity and relates it to structural stability. Let f be a smooth (at least C^1) diffeomorphism of a compact manifold M. A hyperbolic set for f is a closed f-invariant subset $\Lambda \subset M$ such that the tangent bundle of the manifold over Λ splits as a direct sum of two subbundles that are invariant under the derivative, and the derivative of the iterates of the map expands exponentially one of the bundles (the unstable bundle) and contracts exponentially the stable subbundle. These bundles are in general only continuous, but they are integrable. Through each point $x \in \Lambda$, there exists a one-to-one immersed submanifold $W^s(x)$, the stable manifold of x. This submanifold is tangent to the stable bundle at each point of intersection with Λ and is characterized by the fact that the orbit of each point $y \in W^s(x)$ is asymptotic to the orbit of x, and, in fact, the distance between $f^n(y)$ to $f^n(x)$ converges to zero exponentially fast. These are the stable manifolds of Λ. They define a lamination whose holonomy is in general only Hölder continuous (a lamination is a partially defined foliation whose leaves are smooth but do not necessarily fill the whole manifold). However, in dimension two if the diffeomorphism is C^{1^+}, i.e., $C^{1+\alpha}$ for some $\alpha > 0$, the holonomy is also C^{1^+} in the sense that the holonomy mapping between two local transversals extends to a C^{1^+} local diffeomorphism. This property is crucial for the theory developed in the book under discussion. The same holds for the unstable manifolds which are the stable manifolds for the map f^{-1}; see [9]. A *basic set* for f is a hyperbolic set Λ such that it is a maximal invariant subset of a neighborhood U, the periodic orbits are dense in Λ, and there is an orbit which is also dense in Λ. Such a basic set is persistent under perturbations of the diffeomorphism: there exists a neighborhood of $\mathcal{N}$ of f in the space of C^r diffeomorphisms of M endowed with the C^r topology, such that for each $g \in \mathcal{N}$, the maximal invariant set $\Lambda(g)$ in the neighborhood U is hyperbolic and there is a homemorphism $h\colon \Lambda \to \Lambda(g)$ that conjugates f with g. One of the major results in [23] is that if the nonwandering set $\Omega(f)$ is hyperbolic and the periodic points are dense in $\Omega(f)$, then the nonwandering set splits into a finite number of basic sets. Furthermore, if there is no cycle between basic sets of f (a cycle is a periodic sequence of basic sets such that the unstable manifold of each basic set intersects the stable manifold of the next basic set), then f is Ω-stable in the sense that there exists a neighborhood of f such that for each g in this neighborhood there exists a homeomorphism between $\Omega(f)$ and $\Omega(g)$ conjugating f and g.

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A basic set having a neighborhood such that each orbit in the neighborhood is asymptotic to the basic set is called a hyperbolic attractor. A trivial example is an attracting periodic orbit. Another class of examples are the Anosov diffeomorphisms, where the full manifold is a hyperbolic set. A special case is given by a linear map of $\mathbb{R}^n$ that preserves the integer lattice, has determinant one, and has no eigenvalues in the unit circle. It induces an Anosov diffeomorphism on the torus. In this special case the stable manifolds (as well as the unstable manifolds) give a smooth foliation of the torus. Starting from such an Anosov diffeomorphism, with a one-dimensional stable foliation, Smale gave in [23] an important class of examples of hyperbolic attractors called DA (derived from Anosov). The DA is obtained from the Anosov by a surgery that preserves the action on the stable foliation and splits a saddle fixed point into a source and two saddles so that the unstable manifold of the source is a strip bounded by the unstable manifolds of the two saddles and wraps densely in the torus. Its complement in the torus is a hyperbolic attractor. In dimension two this construction can be made symmetric with respect to the orientation reversing involution of the torus that has three fixed points and thus induces an attractor in the quotient of the torus by this involution which is the complement of three points in the sphere. This construction gives a hyperbolic attractor in the two-dimensional sphere that was discovered by Plykin in [18]; see also [7]. These were special cases of a broader class, called *expanding attractors*, that have the property that the unstable manifold of each point is entirely contained in the attractor. The structure of these attractors was analyzed by Williams in [24]. He constructed an appropriate neighborhood of such an attractor and considered the stable plaques, i.e., the connected components of the intersection of the stable manifolds with the neighborhood. Under the hypothesis that the stable foliation is smooth, which is always the case in dimension two, the space of plaques has the structure of a branched manifold, and since the diffeomorphisms maps each plaque into a plaque, it induces an expanding map in the branched manifold. The attractor itself is topologically conjugated to the natural extension of this expanding map and is locally homeomophic to a product of a disc and a Cantor set. In dimension two, except for the Anosov diffeomorphisms, all the attractors are expanding. In fact (see [20]) all of them can be obtained by a construction like the DA starting with a pseudo-Anosov homemorphism that was introduced by Thurston in his classification of the isotopy classes of diffeomorphisms on surfaces in [28]. Thus, the dynamics of such attractors can also be described using train-tracks, as developed in [17].

To study ergodic properties of hyperbolic dynamical systems, Sinai introduced in [21] the important notion of Markov partitions for Anosov diffeomorphisms. This allows the introduction of symbolic dynamics to construct many interesting invariant measures, in particular the so called Gibbs measures. These ideas were extended by Bowen in [2] to the case of hyperbolic basic sets.

The stable and unstable manifolds of hyperbolic sets of diffeomorphisms on two-dimensional manifolds have $C^{1+\alpha}$ holonomy which allows one to use techniques from one-dimensional expanding maps to analyze the fine scale structure of these sets. This is the main theme of the book and involves the notions of rigidity and flexibility of dynamical systems and renormalization. These concepts already played a key role in the development of low-dimensional dynamics in the last 30 years.

The phenomenon of rigidity occurs in many situations when a weaker equivalence between certain classes of dynamical systems automatically implies a stronger equivalence. One of the most celebrated results in this direction is Mostow's rigidity theorem stating that if two compact hyperbolic manifolds are homeomorphic (or even if they have the same homotopy type), they are isometric. This is related to the dynamics of Kleinian groups.

In one-dimensional dynamics the first manifestation of this phenomenon appears in the work of V. Arnold [1], where he proved that a smooth circle diffeomorphism, whose rotation number is irrational satisfying a Diophantine condition and is also close to a rigid rotation, is smothly conjugate to a rotation. The fundamental results of M. Herman, [8] and J-C. Yoccoz, [29] extend this local result by proving that for two smooth circle diffeomorphisms that are topologically conjugate and have a rotation number satisfying a Diophantine condition, the conjugacy is smooth.

Also the discovery by Feigenbaum [5] and Coullet-Tresser [4] of some universal scaling laws in the transition to chaos in parametrized families of interval maps is related to a similar type of rigidity, both in phase space and in the parameter space. A conjecture formulated in order to explain these discoveries involved the dynamics of an infinite-dimension nonlinear operator, the renormalization operator, that attracted the attention of many leading mathematicians like Sullivan, McMullen, Lyubich and others; see [26], [27], [11], [13], [6], [14]. The renormalization operator is defined in certain space of unimodal maps, that is, interval maps that have a unique critical point of quadratic type. The domain of the operator is the set of mappings with the property that there exists a smaller interval around the critical point such that the first return map to this interval is an iterate of the mapping and this iterate restricted to the smaller interval is again unimodal. So the image of the original map by the operator is conjugate to this first return map by an affine map. It follows from many of the articles mentioned above that the dynamics of this operator is very rich and its limit set also has a hyperbolic structure, and the dynamics of maps on its limit set describe the small scale structure of the dynamics of any infinitely renormalizable unimodal map. It follows that, for two such maps that are topologically conjugate, the restriction of the conjugacy to the attractor, which is a Cantor set containing the critical point, extends to a smooth diffeomorphism of the real line.

The same phenomenon was detected in the critical circle mappings which are in the boundary between circle diffeomorphisms and chaotic circle mappings. The rigidity results for one-interval unimodal maps and critical circle maps are related to renormalization.

Sinai and Khanin proved in [22] that the rigidity results of Herman and Yoccoz are also related to renormalization, which is a main theme in the book under review. Finally a rigidity phenomenon was discovered by Shub and Sullivan [25] in the dynamics of expanding maps of the circle: if two such maps are conjugate by an absolutely continuous homeomorphism, then the conjugacy is smooth. A related result was obtained by de la Llave and Moryon in [10]: if the conjugacy between two Anosov diffeomorphisms of the two-dimensional torus maps periodic points into periodic points with the same eigenvalues, then the conjugacy is smooth. A better description of the smooth conjugacy classes of Anosov diffeomorphisms on the two-dimensional torus was obtained in [3].

The book under review is based on a series of articles by the authors and is aimed at experts in the field. The theorems are clearly stated and complete proofs are provided.

Chapter 1, the introduction, describes the main concepts of hyperbolic dynamical systems as discussed above.

In Chapter 2 the affine structures on stable and unstable leaves of Anosov diffeomorphisms by Ghys and Sullivan is extended to more general hyperbolic basic sets using the Hölder ratio structures. It contains the construction of moduli spaces for hyperbolic sets of surface diffeomorphisms describing different smooth realizations of a given topological model.

In Chapter 3 the moduli space of C^{1^+} conjugacies of hyperbolic basic sets is described in terms of a pair of solenoidal functions. These functions were inspired in the scaling functions introduced by Feigenbaum and Sullivan in the setting of one-dimensional dynamics.

In Chapter 4 the structure of the train-track is introduced and, since the holonomies are C^{1^+}, it is shown that they induce smooth maps in the train-tracks that give rise to self-renormalizable structures. The main result is the existence of a one-to-one correspondence between C^{1+} conjugacy classes of hyperbolic diffeomorphisms and pairs of C^{1^+} self-renormalizable structures. As a corollary, it follows that if the conjugacy between two hyperbolic diffeomorphisms is smooth at a point and the derivative has a nonzero determinant, then it is smooth everywhere; i.e., it extends to a smooth map in a neighborhood of the basic set.

In Chapter 5 some rigidity results, obtained initially by Ghys and others for Anosov diffeomorphisms, are extended to hyperbolic basic sets: if the holonomies are smooth enough, then the hyperbolic diffeomorphism is smoothly conjugate to an affine model.

Chapter 6 deals with the existence and uniqueness of Gibbs states for hyperbolic basic sets combining the ideas developed previously by Bowen with the approach of Paterson in the context of limit sets of Fuchsian groups and extended by Sullivan to construct conformal measures of Julia sets.

Chapters 7, 8, 9, and 10 are also dedicated to developing further properties of Gibbs measures with the goal of determining which Gibbs measures are realized by hyperbolic diffeomorphisms and developing the construction of hyperbolic diffeomorphisms with some geometric invariant measures, i.e., invariant measures in a hyperbolic basic set that is absolutely continuous with respect to the Hausdorff measure.

Chapter 11 generalizes to hyperbolic basic sets the result by Livsic-Sinai that gives a formula for the eigenvalues of periodic points of an Anosov diffeomorphism that preserves a measure which is absolutely continuous with respect to Lebesgue measure. For hyperbolic basic sets, a similar formula, that now involves the stable and unstable Hausdorff dimension of the basic set, is proved for diffeomorphisms that preserve a measure absolutely continuous with respect to the Hausdorff measure on the basic set.

In Chapter 12 some ideas from renormalization theory are used to characterize the Lipshitz conjugacy classes of hyperbolic diffeomorphisms that admit an invariant measure which is absolutely continuous with respect to the Hausdorff measure of a basic set.

In Chapter 13 a one-to-one correspondence is established between smooth conjugacy classes of Anosov diffeomorphisms with an invariant measure absolutely continuous with respect to Lebesgue measure and certain tilings of the real line.

Chapter 14 has a detailed discussion on pseudo-Anosov diffeomorphisms.

The five appendices, which cover a substantial part of the book, contain more technical details, and various concepts and results of Sullivan, Pinto, and Rand for one-dimensional dynamics are extended to two dimensions.

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