BOOK REVIEWS

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Theory of Sobolev multipliers: with applications to differential and integral operators, by Vladimir G. Maz'ya and Tatyana O. Shaposhnikova, Grundlehren der mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 337, Springer-Verlag, Berlin, 2009, xiv+614 pp., hardcover, $139.00, ISBN 978-3-540-69490-8

This monumental book, to which for brevity we will be referring to as *Sobolev Multipliers*, was published by Springer in the prestigious series Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. It is devoted to a broadly developed theory of pointwise multipliers (or more precisely, multiplication operators) acting in pairs of Sobolev spaces, as well as other spaces of differentiable functions, along with their applications in analysis, partial differential equations, and mathematical physics. To distinguish pointwise multipliers from well-studied Fourier multipliers and to emphasize the underlying class of function spaces, they are conveniently called Sobolev multipliers.

The book under review consists of two parts. Part I presents a general theory of multipliers in pairs of Sobolev spaces and their generalizations, while Part II is concerned with applications to the Schrödinger operator and its relativistic counerpart, as well as to studies of solutions of certain elliptic partial differential equations, both linear and quasi-linear, in divergence and nondivergence form, along with related pseudo-differential and integral equations. Questions of optimal regularity of the boundary, which are characterized completely in many cases in multiplier terms, as well as the single- and double-layer potential theory, including higher regularity, in Lipschitz domains, and calculus of singular integral operators with symbols in multiplier spaces compose a substantial part of this study.

It is worth mentioning that *Sobolev Multipliers* includes most of the material of the previous book by the same authors entitled *Theory of Multipliers in Spaces of Differential Functions* (Pitman, 1985). It was reviewed by David R. Adams in the *Bulletin of the American Mathematical Society*, **15**, No. 2 (1986), 254–259. That material has been vastly expanded and modernized, both in the scope and methods of the theory, and many new applications have been considered. (The only subject omitted in the present book is multipliers in spaces of analytic functions, which would require a substantial expansion in volume.)

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Multiplicative properties of functions in Sobolev spaces are notoriously hard to investigate. Although concrete examples of multipliers acting in a Sobolev space $W^{k,p}$ had been known for a long time, a study of pointwise multipliers in spaces of differentiable functions was started in earnest by Devinatz and Hirshman [DH], [Hir] (for $p = 2$, disguised as Fourier multipliers of l^2 spaces with a power weight) in the late 1950s and early 1960s, and it was continued by Peetre [P], Polking [Pol], and Strichartz [Str] in the 1960–70s. In particular, in the latter paper ([Str]) a complete characterization was found for multipliers acting in the space of Bessel potentials $H^{k,p}(\mathbb{R}^n)$ in the simpler case $kp > n$.

A systematic study of pointwise multipliers in pairs of Sobolev spaces, including their applications to various problems of analysis and partial differential equations, was originated by Maz'ya and Shaposhnikova. In the 1970–80s, they obtained for the first time necessary and sufficient conditions for a function to be a pointwise multiplier from a Sobolev space $W^{m,p}$ to $W^{k,p}$, where $m, k > 0$ and $1 \le p < \infty$, as well as their analogues for Besov spaces and spaces of Bessel potentials. Their results were presented in detail in their book mentioned above. This work has been fully developed by now into a comprehensive theory with rich connections to other areas of mathematics and its applications in such diverse areas as analysis on manifolds and metric spaces, differential geometry, mathematical physics, and boundary value problems for linear and nonlinear partial differential equations.

The general theory as presented in *Sobolev Multipliers* relies upon the so-called trace inequalities. (This term was coined by Elias M. Stein in the context of integral inequalities for traces of Bessel potentials on linear manifolds.) Trace inequalities are fundamental to understanding multipliers acting from Sobolev spaces to Lebesgue spaces.

A typical trace inequality can be stated in the form

$$\left(\int_\Omega |u|^q \, d\mu\right)^{\frac{1}{q}} \le C \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}, \tag{1}$$

where a constant C does not depend on $u \in C_0^\infty(\Omega)$, and μ is a positive Borel measure on an open set $\Omega \subseteq \mathbb{R}^n$.

In other words the question is, for which measures μ does the embedding $L_0^{1,p}(\Omega) \subset L^q(\Omega, d\mu)$ hold? Here $L_0^{1,p}(\Omega)$ is a homogeneous Sobolev space (denoted by w_p^1 in the book under review) which is defined as the closure of $C_0^\infty(\Omega)$ with respect to the Dirichlet norm $||u||_{L^{1,p}} = \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}}$. The special case where μ is Lebesgue measure and $q = \frac{np}{n-p}$ gives the classical Sobolev inequality when $1 < p < n$. (The endpoint case $p = 1$ is the Gagliardo–Nirenberg inequality.)

Similar questions are of interest for Sobolev spaces $L^{m,p}(\Omega)$ of order m, as well as their inhomogeneous counterparts, $W^{m,p}(\Omega)$, and various fractional order analogues, for instance Besov, Bessel potential, and Triebel–Lizorkin spaces. Recently, similar trace inequalities have been extensively studied on manifolds and metric spaces.

One of the main problems is to characterize completely measures μ so that (1) holds. Such characterizations, especially in the important "diagonal" case $p = q$, require the notion of capacity: for a compact set $E \subset \Omega$, the corresponding $(1, p)$-capacity $\mathrm{cap}_{1,p}(E)$ associated with the right-hand side norm in (1) is defined by

$$\text{cap}_{1,p}(E) = \inf \left\{ \|\nabla u\|^p_{L^p(\Omega)} :\ u \geq 1 \text{ on } \Omega,\ u \in C_0^\infty(\Omega) \right\}. \tag{2}$$

Then it can be shown that (1) holds if and only if

$$\mu(E) \leq c\ \text{cap}_{1,p}(E)^{\frac{q}{p}}, \tag{3}$$

where c does not depend on the compact set $E \subset \Omega$, and $p \geq q > 1$. The class of measures μ obeying (3) was studied extensively in [M3] (see also *Sobolev Multipliers*, Section 1.2). Such measures μ are referred to as Maz'ya measures. They turned out to be very useful in numerous applications to linear and nonlinear partial differential equations.

The proofs of trace inequalities usually make use of the so-called strong capacitary inequality established by Vladimir Maz'ya [M1], [M2] in the early 1960s, and subsequently extended by Maz'ya himself [M3], as well as David R. Adams [Ad], Björn Dahlberg [Dah], Kurt Hansson [Han], and others.

In particular, trace inequalities in the case $d\mu = |f|^q\, dx$, where $f \in L^q_{\text{loc}}(\Omega)$, immediately yield a characterization of multipliers f (or, more precisely, multiplication operators $M_f u = f \cdot u$) acting from $L_0^{1,p}(\Omega)$ (or a more general space of differentiable functions) to a Lebesgue space $L^q(\Omega)$ (with respect to Lebesgue measure dx). Namely, $M_f : L_0^{1,p}(\Omega) \to L^q(\Omega)$ is a bounded operator if and only if

$$\int_E |f|^q\, dx \leq c\ \text{cap}_{1,p}(E)^{\frac{q}{p}}, \tag{4}$$

where c does not depend on the compact set $E \subset \Omega$, and $p \geq q$.

There are other equivalent characterizations of Maz'ya measures, and hence bounded multiplication operators acting from a Sobolev space into a Lebesgue space, especially in the case $\Omega = \mathbb{R}^n$, or under stringent restrictions on the boundary of Ω, which avoid using capacities. Instead, they use notions of energy, nonlinear potentials, or discrete Carleson measures (see, e.g., [KS], [V]). However, these alternative characterizations are not very useful in attacking more challenging multiplier problems since they are more cumbersome, and nonlinear in μ, contrary to (3). There are a number of more convenient sufficient conditions for Maz'ya measures, e.g., the Fefferman–Phong condition [F] and its extension by Chang–Wilson–Wolff [CWW] (here μ must be absolutely continuous with respect to Lebesgue measure), the uniform boundedness of the Wolff potential $\mathbf{W}^{1,p}\mu$ [AH], and their extensions (see [V]). Unfortunately, these conditions are not close to being necessary.

The main goal of *Sobolev Multipliers* is to present complete characterizations and applications of multipliers M_f acting from a Sobolev space $L^{m,p}(\Omega)$ into $L^{k,q}(\Omega)$. The corresponding class of multipliers f is denoted by $M(L^{m,p} \to L^{k,q})$. Similar questions are of interest for pairs of inhomogeneous Sobolev spaces $W^{m,p}$, Besov spaces $B^{m,p}$, etc. The problem becomes much more technical than in the special case $k = 0$ where it reduces to a trace inequality.

The most challenging case turned out to be $m > 0$ and $k < 0$ where multipliers are generally distributions. In particular, when $q = p = 2$, and $m = 1$, $k = -1$, the question of characterizing $M(W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega))$ is equivalent to the form boundedness problem [RS] associated with the Schrödinger operator $\mathcal{H} = \triangle - f$ on

$L^2(\Omega)$. The form boundedness of $\mathcal{H}$ can be expressed in the form of the inequality

$$(5) \qquad |\langle f \cdot u,\, u\rangle| = \left| \int_\Omega |u|^2\, f \right| \le a\, ||\nabla u||^2_{L^2(\Omega)} + b\, ||u||^2_{L^2(\Omega)}, \qquad u \in C_0^\infty(\Omega),$$

for some positive constants a and b, where a is called the form bound. If a can be chosen arbitrarily small for some $b = C(a)$, then $\mathcal{H}$ is said to be infinitesimally form bounded relative to the Laplacian. These notions originate in the so-called KLMN theorem; there are also related notions of relative compactness and p-subordination of sesquilinear forms (see [RS], Section X.2; [RSS], Section 20.4).

The above form boundedness problems, solved only recently for arbitrary distributions $f \in \mathcal{D}'(\Omega)$, are considered in detail in Section 11 of *Sobolev Multipliers*. In fact, a complete characterization of the relative form boundedness of the general second-order differential operator

$$(6) \qquad \mathcal{L} = \sum_{i,\, j=1}^{n} a_{ij}\, \partial_i \partial_j + \sum_{j=1}^{n} b_j\, \partial_j + c : \, W^{1,2}(\mathbb{R}^n) \to W^{-1,2}(\mathbb{R}^n)$$

(not necessarily elliptic, with arbitrary distributional coefficients) was given in [MV] (see *Sobolev Multipliers*, Section 11.6). An analogous form boundedness for the relativistic Schrödinger operator is related to the multiplier class $M(W^{\frac{1}{2},2} \to W^{-\frac{1}{2},2})$ (see *Sobolev Multipliers*, Section 12).

The fundamentals of the theory outlined above form the core of Part I of *Sobolev Multipliers*, Sections 2-5. It includes a thorough study of the endpoint cases $p = 1$ and $p = \infty$, that is, elements of $M(B^{m,1} \to B^{k,1})$, and pointwise multipliers of BMO and Triebel-Lizorkin spaces. Most of this material can be found in *Theory of Multipliers in Spaces of Differential Functions*.

Further developments considered in Part I include a study of the spectrum of a multiplier operator (Section 3), the composition operator on multiplier spaces (Section 4), pointwise interpolation inequalities, and Banach algebras of multipliers in $M(W^{m,p} \to W^{k,p})$ (Section 6), compactness and estimates of the essential norm (Section 7), traces and extension theorems for multiplier spaces (Section 8), change of variables, multiplier manifolds, and implicit function theorems for Sobolev spaces (Section 9). Many results here are sharp, since these problems admit natural answers precisely in terms of multipliers, which provide an appropriate framework for such questions. Thus, the approach developed in *Sobolev Multipliers* changes a standard viewpoint in these matters, and fills a substantial gap in the immense literature on Sobolev spaces and their applications.

Part II of *Sobolev Multipliers* contains a number of significant applications. Besides the problems related to the Schrödinger operator and its relativistic counterpart mentioned above, it covers differential and pseudo-differential operators with coefficients in multiplier spaces (which again provide a natural setting in many instances), solvability and regularity properties of solutions to elliptic boundary value problems with singular coefficients and rough boundaries, various a priori L^p-estimates, as well as applications of multipliers in the single and double layer potential theory, in developing a calculus of singular integral operators whose symbols are multipliers, etc.

The book under review is so extensive and thorough that it would be impossible to describe even briefly many valuable topics covered there as well as their connections with other areas of mathematics. It complements the highly cited books *Sobolev Spaces* by V. G. Maz'ya [M3], now a classic (a new expanded edition is in press), and *Function Spaces and Potential Theory*, by D. R. Adams and L. I. Hedberg [AH], both published by Springer, in 1985 and 1996, respectively. The overlap between them and the present book is not significant.

Sobolev Multipliers is by no means easy reading. Although some of the material is quite technical due to the generality of the problems and the quest for necessary and sufficient conditions, it will be of interest to a great variety of serious readers, from graduate students to experts in analysis, linear and nonlinear differential, pseudo-differential and integral equations, as well as mathematical physicists, differential geometers, and applied mathematicians. It will serve as a valuable reference and guide to the literature, as well as a unique collection of methods and results—indispensable for anyone who is using Sobolev spaces in their work.

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