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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
PIOTR T. CHRUŚCIEL, GREGORY J. GALLOWAY, AND DANIEL POLLACK

MR1316662 (95k:83006) 83C05; 35Q75, 58G16, 83C35

Christodoulou, Demetrios; Klainerman, Sergiu

The global nonlinear stability of the Minkowski space. (English)

Princeton Mathematical Series, 41.

Princeton University Press, *Princeton, NJ*, 1993. x+514 *pp*. $69.50.

ISBN 0-691-08777-6

This book presents the authors' theorem on the stability of Minkowski space, a landmark in the development of mathematical relativity. The book is quite self-contained but it is worth mentioning two useful sources of background information. An article by J.-P. Bourguignon [Astérisque No. 201–203 (1991), Exp. No. 740, 321–358 (1992);MR1157847 (93d:58164)] provides an introduction to various geometrical aspects of the proof while a paper of the authors [Comm. Pure Appl. Math. **43** (1990), no. 2, 137–199;MR1038141 (91a:58202)] shows some of the central analytic tools at work in a simpler setting. The main statement of the theorem is, informally, that, given any initial data set for the vacuum Einstein equations which is sufficiently close to the initial data induced on a hyperplane in Minkowski space, there exists a corresponding solution which is global in the sense of being geodesically complete, and whose asymptotic structure resembles that of Minkowski space. In the book there are three statements of versions of the main theorem which are increasingly precise (and technical). The first two are contained in Chapter 1 (the introduction) while the third is contained in Chapter 10, where the highest level steps of the proof are carried out. The book is not easy to read, due to the very technical nature of its contents, but under the circumstances the quality of the exposition is excellent.

It is impossible to give a useful idea of the proof in this review but some elements of its structure will be used to organize the description of the contents of the various chapters which follows. Energy estimates are the motor which drives the machinery of the proof. They are obtained using the Bel-Robinson tensor, a fourth rank tensor quadratic in the curvature. This tensor is such that, given any timelike conformal Killing vector field in a spacetime, it provides a conservation law which represents a weighted L^2 estimate for the curvature. Commuting with vector fields then gives estimates for derivatives of the curvature. (In the context of this proof this step requires a great deal of care.) This part of the proof is carried out in Chapters 7 and 8. Since the proof deals with arbitrary small perturbations of Minkowski space, in general the spacetimes considered possess no Killing vectors. In fact what is used is that these spacetimes do have approximate Killing vectors, which correspond to the exact Killing vectors of Minkowski space. Combining these approximate Killing vectors with the Bel-Robinson tensor leads to approximate conservation laws. The construction of approximate Killing vectors which give useful estimates is very delicate. This construction, and the estimates it gives rise to, forms the content of Chapters 9 and 11–16. As is clear from what has already been said, the primary estimates for the geometry which are obtained are estimates for the curvature. In order to turn this information into bounds for the metric a number of theorems on elliptic equations on two- and three-dimensional Riemannian manifolds are required. These are proved in Chapters 2–6. In Chapter 10 all the estimates mentioned up to now are linked together, revealing the structure of the proof as an enormous bootstrap argument.

The final chapter of the book is devoted to a closer examination of the asymptotic structure of the spacetimes whose existence is asserted by the theorem. It is shown that the laws of gravitational radiation discovered by Bondi and others more than thirty years ago using formal power series expansions are rigorously true in this class of spacetimes. Thus the reader is given some idea of the question which motivated the study of the problem solved in this book.

Alan D. Rendall

MR2003646 (2004h:83017) 83C15; 83-02, 83C20

Stephani, Hans; Kramer, Dietrich; MacCallum, Malcolm; Hoenselaers, Cornelius; Herlt, Eduard

Exact solutions of Einstein's field equations. (English)

Cambridge Monographs on Mathematical Physics.

Cambridge University Press, *Cambridge*, 2003. xxx+701 *pp*. $120.00.

ISBN 0-521-46136-7

This is the second edition of a book that is very familiar to those whose interests include exact solutions of general relativity. The first edition, published in 1980 [MR0614593 (82h:83002)], quickly became the standard reference work on the subject. As a thorough review of known solutions of Einstein's equations, this first edition also identified many areas in which complete families of exact solutions were not known. Thus it stimulated many research programmes in the following years. As these topics have been addressed, there have been regular demands that a second edition be produced in which all the additional results that have accumulated over the last 20 years are incorporated. After considering over 4000 new papers, the authors have now produced this most valuable second edition.

It may be noted that the first edition appeared at exactly the time when solution-generating techniques were being developed for solving nonlinear equations, and these new methods were being applied to certain symmetry reductions of Einstein's equations. This work has had a profound effect on the subject matter of this book, both in terms of the techniques that can be used and the number of additional solutions to be catalogued. It was partly to ensure a thorough coverage of this material in this new edition that Hoenselaers has been included as another joint author.

In fact, all sections of the first edition have been substantially revised and updated. Some parts have been completely rewritten, and five new chapters have been included. The result is a comprehensive review of known solutions that have been published up to the year 2000.

The book retains the structure of the first edition, in which the first part is devoted to general methods that are used in determining and classifying solutions of Einstein's equations. These chapters form excellent summaries of the relevant topics and will continue to be recommended introductions in their own rights. The most significant change here is the inclusion of two new chapters, on "Invariants and the characterization of geometries" and on "Generation techniques". These are very useful chapters which review topics that have been clarified since 1980. The first addresses the problem of determining the local equivalence of two metrics and how particular space-times can be invariantly classified. The second surveys general methods for generating solutions of nonlinear partial differential equations with comments on how these apply in general relativity.

Part II classifies space-time metrics according to the maximal group of motions or homotheties, its algebraic structure, and the nature and dimension of its orbits. (The recognition of the importance of homothetic motions is a welcome new feature that is included throughout this edition.) As before, there are separate chapters covering topics for which there is an extensive literature, either because of their physical importance, or because they are mathematically tractable and interesting. Most of the chapters in this part have been significantly updated to include recent work. For this edition, new chapters have been included in this part on "Inhomogeneous perfect fluid solutions with symmetry" and "Collision of plane waves". Some sections have also been substantially rewritten and the discussion of a number of topics relocated. In particular, it may be noted that many more perfect fluid solutions have been found in recent years and all these are appropriately classified.

In Part III, algebraically special solutions are presented according to the Petrov-Penrose classification of their Weyl tensor and the geometrical properties of a congruence that is everywhere tangent to the repeated principal null direction. According to standard theorems for this class of solutions, this congruence is normally geodesic and shear-free. Separate chapters treat the cases in which it is twisting or twist-free with nonzero or zero expansion. Although all chapters in this part have been updated to include recent results, it is noticeable that there are few substantial changes here.

Part IV is devoted to the special methods that have been developed for constructing solutions of Einstein's equations. For space-times with two Killing vectors, various solution-generating techniques are now available and the chapter dealing with these has been substantially rewritten. This provides an excellent brief review of the techniques involved and the solutions that have been obtained using them. The chapters on the embedding of four-dimensional Riemannian manifolds and on special vector and tensor fields have been revised but, apart from reviewing recent work on collineations and conformal motion, these are not substantially altered. However, a new chapter, on "Solutions with special subspaces", has been included.

In Parts II, III and IV, exact solutions are classified according to different and independent schemes. The book concludes with a brief final part comprising tables which illustrate the relations between the classification of families of solutions according to these different schemes.

Of course, to keep the size of the book within reasonable limits, much discrimination has been exercised. In particular, the authors have wisely restricted the sources of the solutions considered to the vacuum case and to nonvacuum solutions with electromagnetic, pure radiation or perfect fluid energy-momentum tensors. A nonzero cosmological constant is also occasionally included. Also, since many solutions have been rediscovered numerous times, citations are mostly only given to the earliest reference, papers containing significant advances and reviews. Many references that were included in the first edition are omitted but, with the inclusion of recent papers, over 1,800 publications are now cited. More importantly, although some solutions with symmetries are clearly motivated by physical situations, there is generally very little discussion of the physical interpretation of solutions. Thus, papers that are particularly valuable for the interpretation of solutions are generally not cited. For example, even for the Kerr solution and the C-metric, there is no discussion of the character of their horizons or their global structure. Nowhere is there mention of conformal diagrams. A series of companion volumes on the interpretation of the solutions would therefore be welcome.

This is clearly a most valuable reference book. It comprehensively reviews known local solutions of Einstein's equation and provides a secure base for future research.

J. B. Griffiths

MR2391586 (2008m:83008) 83C05; 35Q75, 58J45, 83-02

Christodoulou, Demetrios

Mathematical problems of general relativity. I. (English)

Zurich Lectures in Advanced Mathematics.

European Mathematical Society (EMS), *Zurich*, 2008. x+147 *pp*.

ISBN 978-3-03719-005-0

This book is based on lecture notes of the author on general relativity. It has three main parts. Chapter 2 introduces some basic ideas about general relativity and the Cauchy problem for the Einstein equations. Conservation laws for isolated systems are discussed in Chapter 3. Finally, Chapter 4 is an account on a somewhat informal and intuitive level of the proof of the nonlinear stability of Minkowski space due to the author and S. Klainerman. Many of the ideas in this book can be found in other texts, but a large number cannot. This review will concentrate on the latter. In addition to the points mentioned specifically below, the text contains many insights on diverse issues, and I believe that even experienced researchers in general relativity will find something new and valuable in the book.

Apart from reviewing the classical local existence proof for the Einstein equations, Chapter 2 brings in original ideas about PDEs derived from a Lagrangian. A more detailed treatment of these can be found in the author's book *The action principle and partial differential equations* [Ann. of Math. Stud., 146, Princeton Univ. Press, Princeton, NJ, 2000;MR1739321 (2003a:58001)]. Chapter 3 contains a treatment of conservation laws in general relativity. As far as the reviewer is aware, nothing comparable is available in the literature. Other accounts are written in a physicist's style which is hard for mathematicians to understand, or assume more background on the Lagrangian or Hamiltonian formalism than many potentially interested mathematicians have. Thus the book makes an important contribution here in providing a transparent exposition. The chapter also contains a largely self-contained proof of the positive mass theorem. The fourth chapter gives a panoramic view of the ideas used in the original proof of the stability of Minkowski space. This book is a valuable addition to the literature on mathematical relativity.

Alan D. Rendall

MR2473363 (2010f:83001) 83-02; 35Q75, 83C05

Choquet-Bruhat, Yvonne

General relativity and the Einstein equations. (English)

Oxford Mathematical Monographs.
Oxford University Press, *Oxford*, 2009. xxvi+785 *pp. ISBN* 978-0-19-923072-3

This massive book covers in detail the areas of classical mathematical relativity theory to which the author has made major contributions during her long and active research career. As the title indicates, the emphasis is on the mathematical properties of the Einstein equations, in particular the local and global existence theorems of the initial value problem. But the inclusion of some introductory chapters on Lorentzian manifolds, special relativity and kinetic theory makes it an up-to-date reference work or even a textbook for an advanced graduate course. (The student would be expected to be familiar with or have access to the well-known book [Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, *Analysis, manifolds and physics*, Second edition, North-Holland, Amsterdam, 1982;MR0685274 (84a:58002); Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, manifolds and physics. II*, North-Holland, Amsterdam, 2000; Zbl 0962.58001].) Parts of classical relativity that are not treated are a systematic study of symmetry groups and other techniques to construct exact solutions of Einstein's equations.

There are 16 chapters totalling more than 500 pages, seven appendices and reprints of some related older papers by the author. A few sections of the main text were contributed by collaborators.

The first five chapters are relatively concise summaries of Lorentzian geometry, special relativity, basic concepts of general relativity, and detailed descriptions of Schwarzschild space-time and of some cosmological models. Chapters six to eight discuss the local initial value problem on spacelike hypersurfaces and the corresponding constraint equations with different gauge choices, the emphasis being on wave (or harmonic) coordinates. There are several different approaches to establish the existence. There follow chapters on the partial differential equations and corresponding initial value problems for relativistic fluids, kinetic theory and progressive waves. The last five chapters describe the present state of understanding global properties of space-times including the notions of causality, singularities and global existence of solutions of initial value problems. Here some very recent theorems (by Christodoulou, Klainerman and others) are carefully described, but proofs could only be sketched since some of these proofs fill whole books by themselves.

The seven appendices provide a very useful summary of the modern techniques for proving existence of solutions of systems of partial differential equations based on Sobolev spaces of tensor fields on Riemannian manifolds. This includes second-order elliptic and different types of hyperbolic systems. Many proofs are included.

Hans-Peter Künzle