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D-modules, perverse sheaves, and representation theory, by Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, translated by Kiyoshi Takeuchi from the 1995 Japanese original, Progress in Mathematics, 236, Birkhaüser Boston, Inc., Boston, 2008, xii+407 pp., US $89.95, ISBN 978-0-8176-4363-8

Overview. The origin of many authors' interest in the connection

$$\text{Representation Theory} \;\rightarrow\; \text{Perverse Sheaves}$$

lies in the solution of the Kazhdan-Lusztig conjecture due to Beilinson-Bernstein [BB] and Brylinski-Kashiwara [BK]. This conjecture expresses coefficients of the Jordan-Hölder series of certain naturally defined infinite-dimensional representations of a semi-simple Lie algebra $\mathfrak{g}$ (Verma modules) in terms of certain explicit polynomials, which, in turn, can be interpreted as coefficients of the Jordan-Hölder series of certain naturally defined perverse sheaves on the flag variety X associated to $\mathfrak{g}$. The link between the coefficients in the Jordan-Hölder series in these two contexts relies on an intermediary object—the category of D-modules; the description of this link realization may be viewed as the goal of the present book.

Why D-modules? The ubiquity of D-modules in modern geometric representation theory can be explained by their two-faceted nature. Given an algebraic variety X, on the one hand, D-modules on it are objects that can be explicitly constructed by generators and relations, the latter being algebraic differential operators acting between vector bundles on X. On the other hand, the category of D-modules on X behaves like a *sheaf theory*; that is, D-modules can be restricted (resp. extended) from open and closed subsets, tensored together, and, more generally, for a map between algebraic varieties $f : X_1 \to X_2$ there are direct and inverse image functors f_* and $f^!$.

D-modules via generators and relations. For a smooth algebraic variety X over a field of characteristic zero, (local) algebraic differential operators form a sheaf of algebras on X, and, by definition, D-modules on X are sheaves of modules over D_X that are quasi-coherent as modules over the structure sheaf $\mathcal{O}_X$. In particular, D_X itself is a D-module.

In addition, given a matrix of differential operators $||d^i_j||$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ we can consider it as a map $T : D_X^{\oplus n} \to D_X^{\oplus m}$, and we can attach to it the D-module $\mathcal{F} := \operatorname{coker}(T)$. For example, when we work over the field of complex numbers, the set of maps of D-modules from $\mathcal{F}$ to the D-module of functions (or distributions) on X of a specified class is in a bijection with the set of solutions of the system of differential equations given by $||d^i_j||$ in this class of functions (or distributions).

Dually, given a D-module $\mathcal{F}$ on X, we can take its global sections $\Gamma(X, \mathcal{F})$ as a quasi-coherent sheaf, and it will be a vector space, acted on by the algebra of global differential operators on X.

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D-modules and representation theory. Here is how the above considerations play out in representation theory. Let X be as above, and let $\mathfrak{g}$ be a Lie algebra that acts on X by vector fields. In particular, we have a map of associative algebras $\alpha : U(\mathfrak{g}) \to \Gamma(X, D_X)$. The constructions described above give rise to a pair of mutually adjoint functors

$$\mathrm{Loc} : \mathfrak{g}\text{-mod} \leftrightarrows D_X\text{-mod} : \Gamma,$$

where Loc is called the localization functor. Explicitly, it sends a $\mathfrak{g}$-module $\mathcal{M}$, given as the cokernel of a map $U(\mathfrak{g})^{\oplus n} \to U(\mathfrak{g})^{\oplus m}$ corresponding to a matrix $||u^i_j||$ of elements of $U(\mathfrak{g})$, to $\mathcal{F}$, corresponding to the matrix of differential operators $||\alpha(u^i_j)||$.

One can wonder how closely related are the categories $\mathfrak{g}$-mod and D_X-mod. Of course, for general X and $\mathfrak{g}$ one would not be able to say much. However, there is one specific case, of particular interest to representation theorists, when it turns out that the two categories above are (almost) equivalent. This is the case when $\mathfrak{g}$ is a semi-simple Lie algebra and $X := G/B$ is its flag variety. Then the statement of the theorem, due to Beilinson-Bernstein is that the above two functors induce mutually inverse equivalences between the category of D-modules on X and that of $\mathfrak{g}$-modules, on which the center of $U(\mathfrak{g})$ acts by the trivial central character. This is the subject of Chapter 11 of the book.

This theorem allows us to express the Jordan-Hölder coefficients for $\mathfrak{g}$-modules that appear in the Kazhdan-Lusztig conjecture in terms of Jordan-Hölder coefficients of some specific D-modules on the flag variety X. The latter D-modules are in fact the "constant" D-modules corresponding to the Schubert cells, i.e., Borel orbits on X. This is explained in Chapters 12 and 13 of the book.

D-modules vs. sheaves. As was mentioned above, the category of D-modules on an algebraic variety X behaves much like the category of (constructible) sheaves; however, there are some substantial differences. Namely, for a map of algebraic varieties $f : X_1 \to X_2$, for sheaves we have two pairs of mutually adjoint functors

$$f^* : \mathrm{Sh}(X_2) \leftrightarrows \mathrm{Sh}(X_1) : f_* \quad \text{and} \quad f_! : \mathrm{Sh}(X_1) \leftrightarrows \mathrm{Sh}(X_2) : f^!,$$

whereas for D-modules, only their right adjoint counterparts, namely f_* and $f^!$ are defined, whereas the left adjoint functors do not exist in general. Therefore, there cannot be any equivalence between the entire category of D-modules on X and that of sheaves. However, the situation can be remedied.

To compare the two sides, we shall need to assume that the ground field is that of complex numbers, and replace the abelian categories D_X-mod and $\mathrm{Sh}(X)$ by their derived categories, denoted $\mathrm{D}(D_X\text{-mod})$ and $\mathrm{D}(\mathrm{Sh}(X))$, respectively. It will turn out that both sides contain (full) subcategories that are actually equivalent. The corresponding subcategory on the right-hand side is easy to describe: it is the subcategory $\mathrm{D}_{\mathrm{constr}}(\mathrm{Sh}(X)) \subset \mathrm{D}(\mathrm{Sh}(X))$ consisting of complexes of sheaves with constructible cohomology (i.e., complexes $\mathcal{S}$ for which there exists a decomposition $X = \bigcup_i X_i$ into locally closed algebraic subvarieties, such that each of the restrictions $\mathcal{S}|_{X_i}$ is locally constant). The situation for D-modules is more technically involved.

Coherent, holonomic, and regular D-modules. The sought-for subcategory inside $\mathrm{D}(D_X\text{-mod})$ is singled out by a three-step procedure. First, one imposes a local finite-generation condition, and obtains the subcategory $(D_X\text{-mod})_{\mathrm{coh}}$ of coherent D-modules, and the corresponding subcategory $\mathrm{D}_{\mathrm{coh}}(D_X\text{-mod}) \subset \mathrm{D}(D_X\text{-mod})$, consisting of complexes, whose cohomologies are coherent.

To an object $\mathcal{F} \subset (D_X\text{-mod})_{\mathrm{coh}}$ one can assign a numerical invariant called the functional dimension (see Section 2.3 of the book). It turns out that for any nonzero $\mathcal{F}$, its functional dimension is $\geq \dim(X)$; for example, for $\mathcal{F} = D_X$, the functional dimension equals $2\dim(X)$. D-modules with the minimal possible functional dimension, i.e., $\dim(X)$, are called holonomic; they form an abelian subcategory denoted $(D_X\text{-mod})_{\mathrm{hol}} \subset (D_X\text{-mod})_{\mathrm{coh}}$. This category has a number of extremely favorable properties, discussed in detail in Chapter 3 of the book; e.g., every object of $(D_X\text{-mod})_{\mathrm{hol}}$ is of finite length. Moreover, for a map $f : X_1 \to X_2$, the functors f_* and $f^!$ preserve holonomicity, and, most importantly, their left adjoints f^* and $f_!$ are defined on the holonomic subcategories. That is, the subcategory $\mathrm{D}_{\mathrm{hol}}(D_X\text{-mod}) \subset \mathrm{D}_{\mathrm{coh}}(D_X\text{-mod})$, consisting of complexes with holonomic cohomologies, does indeed behave a lot like the category of sheaves with constructible cohomology.

However, $\mathrm{D}_{\mathrm{hol}}(D_X\text{-mod})$ itself is not yet equivalent to $\mathrm{D}_{\mathrm{constr}}(\mathrm{Sh}(X))$. One needs to perform one more step, namely, to single out among all holonomic D-modules those that are regular (see Chapter 6 of the book). The idea is that the systems of differential equations $\frac{df}{dx} = f$ and $\frac{dg}{dx} = 0$ are equivalent analytically by means of $f = g \cdot \exp(x)$, but not algebraically. One obtains an abelian subcategory $(D_X\text{-mod})_{\mathrm{reg}} \subset (D_X\text{-mod})_{\mathrm{hol}}$ and the corresponding subcategory $\mathrm{D}_{\mathrm{reg}}(D_X\text{-mod}) \subset \mathrm{D}_{\mathrm{hol}}(D_X\text{-mod})$.

Now, the main theorem, known as Riemann-Hilbert correspondence, due to Kashiwara, states that there exists an equivalence

$$\mathrm{D}_{\mathrm{reg}}(D_X\text{-mod}) \simeq \mathrm{D}_{\mathrm{constr}}(\mathrm{Sh}(X)),$$

compatible with all other operations (i.e., the functors f^*, $f^!$, f_*, $f_!$). This equivalence is discussed in Chapters 4, 7 and 7 of the book.

Finally, one can ask, Can one describe the abelian subcategory of $\mathrm{D}_{\mathrm{constr}}(\mathrm{Sh}(X))$ that corresponds to $(D_X\text{-mod})_{\mathrm{reg}} \subset \mathrm{D}_{\mathrm{reg}}(D_X\text{-mod})$ under Riemann-Hilbert? The naive guess that it is the category of ordinary sheaves with constructible cohomology is not correct. The true answer is that it is the category of perverse sheaves $\mathrm{Perv}(X)$ of perverse sheaves on X, discussed in Chapter 8 of the book.

The book by Hotta, Takeuchi, and Tanisaki. The present book provides a reader-friendly treatment of the subject, suitable for graduate students who wish to enter the area.

Part I of the book presents the theory of D-modules (the review of the theory given above corresponds to the contents of Part I, chapter by chapter). The treatment in the book is quite complete, but rather condensed, and in order to acquire fluency in the subject, the reader may find it useful to combine this book with another text covering the same topics, especially for Chapters 1–3.

Part II provides the necessary background in the structure of semi-simple Lie algebras and their representations.

References

[BB] A. Beilinson and J. Bernstein, *Localization de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math., **292**-1 (1981), 15–18. MR610137 (82k:14015)

[BK] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math., **63**-3 (1981), 387–410. MR632980 (83e:22020)

[KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math., **52**-2 (1979), 165–184. MR560412 (81j:20066)

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