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Simple groups of finite Morley rank, by T. Altinel, A. V. Borovik, and G. Cherlin, Mathematical Surveys and Monographs, 145, American Mathematical Society, Providence, RI, 2008, xix+556 pp., US $109.00, ISBN 978-0-8218-4305-5

The Algebraicity Conjecture treats model-theoretic foundations of algebraic group theory. It states that any simple group of finite Morley rank is an algebraic group over an algebraically closed field. In the mid-1990s a view was consolidated that this project falls into four cases of different flavour: even type, mixed type, odd type, and degenerate type. This book contains a proof of the conjecture in the first two cases, and much more besides: insight into the current state of the other cases (which are very much open), applications for example to permutation groups of finite Morley rank, and open questions. The book will be of interest to both model theorists and group theorists: techniques from the classification of finite simple groups (CFSG), and from other aspects of group theory (e.g., black box groups in computational group theory, and the theory of Tits buildings, especially of generalised polygons) play a major role.

The techniques used are primarily group theoretic, but the history and motivation are more model theoretic. The origins of the conjecture, as with much modern model theory, lie in Morley's Theorem: this states that if T is a complete theory in a countable first order language L (that is, T is a maximally consistent set of L-sentences) and T is κ-categorical for some uncountable cardinal κ, then T is κ-categorical for *all* uncountable κ, that is, T is *uncountably categorical*. Here, the theory T is *κ-categorical* if all its models of size κ are isomorphic. Morley's proof in [8] introduces the notion of *Morley rank*, an abstract dimension notion for definable sets (i.e., solution sets of formulas). In an instructive example, where T is the theory of an algebraically closed field K, definable sets are exactly the same as constructible sets (by quantifier elimination), and the Morley rank of a constructible set is just the algebraic-geometric dimension of its Zariski closure. Morley rank is ordinal valued: the Morley rank of a definable set X is at least $\alpha+1$ if it is possible (at least after moving to an elementary extension) to partition X into infinitely many definable sets of rank at least α, and the definition at limit ordinals is the natural one. Morley showed that in an uncountably categorical theory, all definable sets have ordinal-valued Morley rank. In particular, such theories are *stable* (in fact, ω-stable). The rank was later shown by Baldwin to be finite.

A new proof of Morley's Theorem, yielding additional information, was given by Baldwin and Lachlan [1]. They showed that any uncountably categorical structure is "coordinatised" by a *strongly minimal set*, that is, a definable set all of whose definable subsets are finite or cofinite, uniformly in parameters. The form of coordinatisation was later refined by Zilber in his "Ladder theorem"; he also formulated conjectures on the structure of strongly minimal sets: obvious examples are pure sets (trivial geometry), vector spaces (locally modular geometry) and algebraically closed fields, and Zilber conjectured that any strongly minimal set with nonlocally modular geometry is very close to an algebraically closed field. This was proved false by Hrushovski [6]. With a new and delicate amalgamation technique

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now used widely, he constructed nonlocally modular strongly minimal sets with no connection to fields. Zilber's conjecture on strongly minimal sets, though false in general, has been highly influential. It is true in certain key situations which are used, for example, in Hrushovski's work on the Mordell-Lang and Manin-Mumford conjectures.

Zilber realised early in the 1970s that simple algebraic groups over algebraically closed fields are uncountably categorical, so have finite Morley rank. In fact, any infinite simple group of finite Morley rank is uncountably categorical. Also, in his Ladder theorem, certain key definable groups of permutations are either abelian, or non-abelian simple (and of finite Morley rank). These considerations made the Algebraicity Conjecture both plausible and model-theoretically well motivated; it would have followed from the truth of Zilber's conjecture on strongly minimal sets. The conjecture was probably first posed as a question by Zilber, and was formulated in Cherlin's important paper [5], without the finiteness of rank assumption. Until recently, the conjecture was known as the Cherlin-Zilber Conjecture.

Early results fundamental to the conjecture include Macintyre's structure theory for infinite ω-stable abelian groups and fields (the latter are algebraically closed); identification of a model-theoretic analogue of the *connected component* G° of an algebraic group G—namely, the smallest definable subgroup of finite index; the Baldwin-Saxl paper [2], which gives chain conditions for definable sets in a much more general context, that of stable groups; and methods developed by Baur, Cherlin and Macintyre, giving strong structural results for $\aleph_0$-categorical stable groups. Cherlin's paper [5] builds on these and shows that a connected group of Morley rank at most 2 is soluble, and a rank 3 simple group with a rank 2 definable subgroup is isomorphic to $\mathrm{PSL}_2(K)$, where K is algebraically closed. Here, perhaps, methods from finite group theory were first used in this context. Earlier, Zilber obtained results critical to the modern structure theory, often mimicking techniques from linear algebraic groups: the Indecomposability Theorem, used to prove definability of certain subgroups and in particular to show that any non-abelian group of finite Morley rank with no proper nontrivial *definable* normal subgroup is simple; results showing that solvable and nilpotent subgroups have a "definable closure" with the same properties, analogues of the Lie-Kolchin theorem in algebraic group theory; and techniques for defining a field in a group.

Another method developed in the later 1970s and early 1980s was the use of *generic types* of a group. Even in the broader context of stable theories (or wider contexts, some being investigated at present), generic types of groups play a central role. There are various equivalent definitions. Generic types correspond to cosets of the connected component, so in particular a connected stable group has a unique generic type. In the finite Morley rank case, a generic type of a group (over a model of the theory) is one of maximal Morley rank.

These early results already suggest strongly that groups of finite Morley rank resemble algebraic groups. Morley rank itself resembles Zariski dimension. In books such as [4] and that under review, following Borovik and Poizat, groups of finite Morley rank are introduced as *ranked groups*, i.e., groups with a rank function on definable sets taking values in the natural numbers and satisfying simple and geometrically natural axioms. However, there is no version of the Zariski topology. Also, there are very few results saying that a group has a property if all elements which have generic type have that property; for example, this is open for the property "exponent n", though the book under review has partial results on this.

An important early predecessor to this book was written by Poizat [9]. This was set in a broader context (stable groups), and it includes much background material and some early work of Hrushovski (e.g., on internality and analysability). It also describes the structure of certain putative counterexamples to the Algebraicity Conjecture, namely *bad groups*: simple groups of finite Morley rank all of whose definable connected proper subgroups are nilpotent (Poizat did not assume simplicity). The wider context of stable groups was developed much further, especially by Wagner; see for example his book [10]. I emphasise this wider context: the Algebraicity Conjecture has an easily stated analogous conjecture for stable groups, namely that any infinite simple group with stable theory is the set of K-rational points of an algebraic group defined over a stable field K. In fact, I am not aware of any infinite simple groups which do *not* arise from algebraic groups and which satisfy any reasonable generalisation of stability. For example, is there such a group whose first order theory is *simple* or is *NIP*? (These notions are both due originally to Shelah and are of considerable recent interest.)

The first general classification result towards the Algebraicity Conjecture was Simon Thomas's proof of the conjecture in the locally finite case, resting on the classification of finite simple groups. Later, a different proof, not using CFSG, was found.

Progress gathered pace in the mid-1980s. The special role of the prime 2 emerged in [3], where conjugacy of Sylow 2-subgroups was proved; this remains open for other primes. In fact, the relevant notion is often the Sylow° 2-subgroup, the connected component of the Sylow 2-subgroup. Important results were obtained by A. Nesin, for example on rings of finite Morley rank. Then in a lecture in Durham in 1988, Borovik outlined a strategy for tackling the Algebraicity Conjecture by seeing it as a scaled-down model of CFSG, to be tackled mimicking ideas from there. In certain respects the finite Morley rank case might be much simpler: there may be no analogue of sporadics and twisted groups of Lie type, there are no problems with small fields (for example accidental isomorphisms between groups, and vanishing of 1-dimensional tori), and some techniques from the theory of algebraic groups are available. Borovik's vision, modified with experience, has remained the general approach. The connection to CFSG should go in two directions. Techniques from CFSG should guide the finite Morley rank programme (though new proofs will always be needed); and the programme should give valuable insights into CFSG, including into second- and third-generation proofs.

It seems that there is currently no great confidence in the truth of the Algebraicity Conjecture in all cases. Certain tools available in the finite case, and used in CFSG, are absent: counting arguments, transfer, and character theory. In particular, there is no analogue of the Feit-Thompson Theorem that finite groups of odd order are soluble. Indeed, the degenerate case (showing that there are no simple groups of finite Morley rank with finite—in fact trivial—Sylow 2-subgroup) seems to be out of sight by current methods. A counterexample to the Algebraicity Conjecture may well live in the degenerate case and, indeed, might be torsion-free. Earlier, it was hoped that the Hrushovski construction technique, mentioned above, might provide a counterexample. It has yielded various exotic structures—a strongly minimal set with two independent algebraically closed field structures in different characteristics and disjoint languages, Baudisch's nilpotent class 2 non-locally modular group (Morley rank 2), and "bad fields" (noted below), also of Morley rank 2. But my impression is that people are now looking elsewhere for counterexamples, perhaps to geometric group theory.

For a simple algebraic group over an algebraically closed field K, the Sylow° 2-subgroups are either of bounded exponent (if $\mathrm{char}(K) = 2$) or are divisible (odd or zero characteristic). This suggests a case division for the Algebraicity Conjecture, justified by the conjugacy of the Sylow 2-subgroups. A simple group of finite Morley rank is said to have *even type* if the Sylow° 2-subgroups are infinite and have bounded exponent; it has *odd type* if they are infinite divisible; in *mixed type*, they are a product of an infinite bounded group and an infinite divisible group; and in the *degenerate* case, the 2-Sylow is finite.

During the 1990s, Tent and coauthors obtained results on Tits buildings in the finite Morley rank context, heavily used in even type. Extending work of Nesin and of Baldwin, and using Hrushovski's amalgamation technique mentioned above, Tent showed that for every $n \geq 3$ there is a generalised n-gon of Morley rank 2 whose automorphism group is transitive on ordinary $(n+1)$-gons, so has a *BN pair* in the sense of Tits. Then, in [7], a classification was given of *Moufang* generalised polygons of finite Morley rank; this uses the Tits-Weiss classification of Moufang generalised polygons. From this, the Algebraicity Conjecture is easily verified for infinite simple groups of finite Morley rank with a definable spherical BN pair of rank at least 3.

The state of play in the early 1990s was described in the book [4]. Since then, enormous progress has been made, a major chunk described in the book under review. In the mid-1990s, this subject took on a rather forbidding and technical hue, uninviting to outsiders. For various reasons this has changed, and the subject is now rich and wide ranging, even though the conjecture as a whole remains inaccessible. One reason for this change is the above-mentioned geometric work of Tent and others. Another is model-theoretic input of Wagner on fields of finite Morley rank. Partly through work of Jaligot, the technical assumption "no interpretable bad fields" has generally been removed from results. (A *bad field* is a field of finite Morley rank such that the multiplicative group has a proper infinite definable subgroup.) Progress has been made on bad fields: using Hrushovski amalgamation, bad fields have been constructed in characteristic 0, but in characteristic p, by work of Wagner, their existence would imply that there are just finitely many p-Mersenne primes. Borovik has found use of methods from computational group theory, the idea being that a *random* group element is like a model-theoretically *generic* element. Very importantly, Sela's work on the model theory of free groups, and in particular his result that they (and hyperbolic groups) have stable theory, hints at possible counterexamples in the degenerate case. Borovik and Cherlin have found applications of very recent results to permutation groups of finite Morley rank, and they have raised interesting questions on permutation groups just in the algebraic group context. Hrushovski has suggested an attractive approach to a proof of the Algebraicity Conjecture, at least in certain situations, taking a generic automorphism (if it exists) of a putative counterexample. Under reasonable assumptions the fixed point set of the automorphism, though unstable, will have simple theory and will have a measure on definable sets which may support new counting arguments. There is new work on groups satisfying weaker model-theoretic hypotheses, such as superstability. In the purer part of the subject, Frecon and Jaligot have developed a theory of Carter subgroups, and Burdges has provided aspects of a Sylow theory in primes other than 2.

The book under review gives a proof that any simple group of finite Morley rank of even type is an algebraic group, and that the case of mixed type does not occur. This is a massive achievement. The authors stress that methods of finite group theory provide a variety of routes to it, that choices are inevitably made, and the proof is not canonical. The original goal was apparently to prove these theorems for K^*-groups (groups of finite Morley rank whose proper definable infinite simple sections are Chevalley groups), but the K^*-condition was removed, partly through work of Altinel. For odd type (not treated in detail here), the K^*-assumption remains, and one of the main results is that for a simple K^*-group of finite Morley rank and odd type, either G is algebraic or the Sylow$^\circ$ 2-subgroups have 2-rank at most 2. Initially, for even type there was also an assumption that no bad fields were interpretable, but this was removed through work of Jaligot, and apparently is also less important now in odd type.

The book falls into three parts. Part A develops techniques for the main theorem. This includes a bag of tools from both group theory and model theory (Chapter I). Some of this is classical, but it includes some of Frecon's work from the late 1990s, e.g., on the conjugacy of Carter subgroups in a soluble group of finite Morley rank, and other recent material. Chapter II contains background on algebraic groups, and more generally on K-groups: groups of finite Morley rank whose definable connected simple sections are Chevalley groups. It should be emphasised that the book deals with groups *possibly with extra structure*, i.e., not necessarily parsed in the language of groups, so even for Chevalley groups over an algebraically closed field, model-theoretic phenomena beyond the algebraic category may arise. The remaining chapters in Part A include, along with much else, a useful treatment of buildings of finite Morley rank and in particular Moufang generalised polygons, an identification of a BN-pair via a version of Niles's Theorem, and an identification theorem based on Curtis-Tits and used via signalizer functors. There are also important newish results on conjugacy (of maximal "good tori") and genericity, leading to a proof that in the degenerate case, the Sylow 2-subgroup is trivial (rather than merely finite).

Part B is a single chapter, and it shows that the mixed type case does not occur (at this stage modulo the assumption that the Algebraicity Conjecture holds in even type). The proof seems relatively nontechnical and accessible. It is based on an analysis, for a putative counterexample G, of a graph whose vertices are the nontrivial 2-unipotent subgroups, adjacent if commuting.

Part C contains the proof of the Algebraicity Conjecture in even type. As expected, much of the work (the bulk of Chapters VI–VIII) is devoted to producing identification theorems for SL_2, where there is less structure to work with. At the end of Chapter VIII the "plan of attack" is described, for a simple group G of finite Morley rank and even type. If S is a Sylow$^\circ$ 2-subgroup of G, let $\mathcal{M}(S)$ denote the class of subgroups P of G containing $N_{G^\circ}(S)$ and such that $U_2(P)/O_2(P) \cong \mathrm{SL}_2(K)$ for some characteristic 2 algebraically closed field K (I omit definitions here). Then G is *thin* if $\mathcal{M}(S) = \emptyset$, *quasithin* if there are $P_1, P_2 \in \mathcal{M}(S)$ with $G = \langle U_2(P_1), U_2(P_2)\rangle$, and *generic* otherwise. In the thin case, there is a rapid proof that $G \cong \mathrm{SL}_2(K)$, and in the generic case, Niles's Theorem plus the theory of buildings of finite Morley rank (or alternatively the Curtis-Tits identification method) identifies G as an algebraic group of Lie rank at least three.

There remains the difficult quasithin case, treated in Chapter IX. Here, it is shown that G has Lie rank 2: more precisely, G is isomorphic to one of $\mathrm{PSL}_3(K)$, $\mathrm{G}_2(K)$, $\mathrm{Sp}_4(K)$, where K is algebraically closed of characteristic 2. The proof is based on the amalgam method, part of a third-generation proof of CFSG. One identifies, from subgroups P_1, P_2 "of minimal parabolic type" a graph of groups, whose universal cover (a tree) has a generalised n-gon as a quotient, and then applies [7].

The book concludes with an extensive chapter covering the state of play in the cases of odd and degenerate type, a discussion of applications to permutation groups of finite Morley rank (work of Borovik and Cherlin), discussions of the overall connections to CFSG, and a rich list of open problems.

Great care in the book is paid to exposition. Each of the ten chapters has an extensive and very helpful introduction, together with some concluding historical notes. Material used from elsewhere (such as the work on generalised n-gons) is not always treated in full detail, but essential issues are emphasised.

Potential readers may be put off by the daunting and highly technical material. The Algebraicity Conjecture has been the main and coordinated programme of about ten researchers, with others (such as Poizat, Tent, and Wagner) also making major contributions. It is unlikely that model theorists not working close to the programme will grapple with all the technical details, but there is material in this book to stimulate many model theorists, and it will be essential reading for anyone, such as a graduate student, working on the programme. It could also be of great interest and potential value to researchers in finite and algebraic group theory. Access to the technical core of the book will be easier for them than for model theorists.

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