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*Optimal transport: old and new*, by Cédric Villani, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 338, Springer-Verlag, Berlin, 2009, xxii+973 pp., US \$159.00, ISBN 978-3-540-71049-3

## 1. Introduction

The optimal transport problem has received the attention of many researchers in the last two decades, and its popularity is still increasing. This is mainly motivated by the discovery of unexpected connections between optimal transport and problems in physics, geometry, partial differential equations, etc. To give an example, consider the following geometric statement:

*Let* $(M_k, g_k, \mathrm{vol}_k)$ *be a sequence of smooth compact Riemannian manifolds with nonnegative Ricci curvature, converging in the measured Gromov-Hausdorff sense to a smooth compact Riemannian manifold* $(M_\infty, g_\infty, \mathrm{vol}_\infty)$*. Then* $(M_\infty, g_\infty)$ *has nonnegative Ricci curvature.*

At first sight, this statement may look surprising. Indeed the Gromov-Hausdorff convergence is a very weak notion, so it may seem strange that it can control lower bounds on the Ricci curvature, which a priori should depend on second derivatives of the metric. However, optimal transport allows recasting lower Ricci bounds in terms of much more robust inequalities (see Theorem 4.1 below), and this fact is at the core of the proof of the above result [8, 14, 15].

The present article briefly surveys the exciting and extremely active field of optimal transport, with emphasis on the content and features of the book under review.

## 2. The optimal transport problem

The optimal transport problem (whose origin goes back to Monge [12]) is nowadays formulated in the following general form: given two probability measures $\mu$ and $\nu$ defined on measurable spaces $X$ and $Y$, find a measurable map $T : X \to Y$ with $T_\#\mu = \nu$ (i.e., $\mu(T^{-1}(A)) = \nu(A)$ for any $A \subset Y$ measurable), and in such a way that $T$ minimizes the transportation cost. This last condition means

$$\int_X c(x, T(x))\, d\mu(x) = \min_{S_\sharp\mu=\nu} \left\{ \int_X c(x, S(x))\, d\mu(x) \right\},$$

where $c : X \times Y \to \mathbb{R}$ is a given cost function. When the transport condition $T_\sharp\mu = \nu$ is satisfied, we say that $T$ is a *transport map*, and if $T$ also minimizes the cost, we call it an *optimal transport map*.

The major advance on this problem is due to Kantorovich, who proposed in [5, 6] a notion of weak solution of the optimal transport problem. He suggested looking for *plans* instead of transport maps, that is probability measures $\gamma$ in $X \times Y$ whose first and second marginal are $\mu$ and $\nu$, respectively. Denoting by $\Pi(\mu, \nu)$ the set of

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plans, the new minimization problem becomes

$$\min_{\gamma\in\Pi(\mu,\nu)}\left\{\int_{X\times Y} c(x,y)\,d\gamma(x,y)\right\}. \tag{1}$$

The problem of the existence and uniqueness of optimal maps when the cost is the distance squared is now well understood: there are classical results by Brenier in Euclidean spaces [1, 2] and by McCann on (compact) Riemannian manifolds [11]. Roughly speaking, these results state that if the source measure is absolutely continuous with respect to the Lebesgue (resp. volume) measure, then the optimal transport map exists and is unique. However, let us point out that, for many applications, not only it is important to know the existence and uniqueness of an optimal map but also to have some information on its structure. For instance, in the Euclidean case the optimal map is given by the gradient of a convex function, while on a Riemannian manifold there exists a semiconvex function $\varphi$ such that $T(x) = \exp_x(\nabla\varphi(x))$.

In Part 1 of the book under review, the author studies the problem of existence and uniqueness for optimal maps in great generality, showing general sufficient conditions on the cost functions which ensure that the problem has a unique solution, when the source measure $\mu$ is absolutely continuous.

## 3. The Monge-Ampère equation and regularity of optimal maps

To explain the link between optimal transport and the Monge-Ampère equation, let us consider the case $\mu(dx) = f(x)\mathrm{vol}(dx)$ and $\nu(dy) = g(y)\mathrm{vol}(dy)$ on Riemannian manifolds, with cost given by $c(x,y) = \frac{1}{2}d(x,y)^2$. The condition $T_\#\mu = \nu$ formally gives

$$|\det(DT(x))| = \frac{f(x)}{g(T(x))}.$$

Exploiting the relation $T(x) = \exp_x(\nabla\varphi(x))$ recalled above, the above equation becomes a Monge-Ampère type equation for $\varphi$, which takes the form

$$\det\big(D^2\varphi(x) + A(x,\nabla\varphi(x))\big) = h(x,\nabla\varphi(x)), \tag{2}$$

where $A(x,\nabla\varphi(x)) = \nabla_x^2 c\big(x,\exp_x\big(\nabla\varphi(x)\big)\big)$ and $h = h(x,p)$ depends on $f$, $g$, and the cost $c$. Assume $f$ and $g$ to be $C^\infty$ and strictly positive on $M$. A natural question is whether the optimal map $T$ is smooth or not.

In the case $M = \mathbb{R}^n$, the above equation reduces to the classical Monge-Ampère

$$\det(D^2\phi(x)) = \frac{f(x)}{g(\nabla\phi(x))}, \qquad \phi(x) = \varphi(x) + \frac{|x|^2}{2}, \quad T(x) = \nabla\phi(x).$$

This problem has been solved by Caffarelli [3], who showed that the convexity of the support of $g$ is the natural geometric condition needed to prove the global smoothness of $\phi$ (and thus of $T$), when $f$ and $g$ are smooth and bounded away from zero on their respective support.

In the general case, the presence of the term $\nabla_x^2 c\big(x,\exp_x\big(\nabla\varphi(x)\big)\big)$ in (2) can create obstructions to the smoothness. In [9], the authors found a fourth-order condition on the cost function, which turned out to be a sufficient and necessary condition to prove regularity results. The idea was to differentiate (2) twice in order to get a linear PDE for the second derivatives of $\varphi$, and then try to show an a priori estimate on the $L^\infty$-norm of $D^2\varphi$. In this computation, one ends up at a certain stage with a term which needs to have a sign in order to conclude the argument.

This term is what now is called the Ma-Trudinger-Wang tensor (in short MTW tensor):

$$\mathfrak{S}_{(x,y)}(\xi,\eta) := \frac{3}{2}\sum_{ijklrs}\left(c_{ij,r}c^{r,s}c_{s,kl} - c_{ij,kl}\right)\xi^i\xi^j\eta^k\eta^l, \qquad \xi \in T_xM,\ \eta \in T_yM.$$

In the above formula the cost function is evaluated at $(x,y)$, and we used the notation $c_j = \frac{\partial c}{\partial x^j}$, $c_{jk} = \frac{\partial^2 c}{\partial x^j \partial x^k}$, $c_{i,j} = \frac{\partial^2 c}{\partial x^i \partial y^j}$, $c^{i,j} = (c_{i,j})^{-1}$, and so on. The condition to impose on $\mathfrak{S}_{(x,y)}(\xi,\eta)$ is $\mathfrak{S}_{(x,y)}(\xi,\eta) \geq 0$ whenever $\sum_{ij} c_{i,j}\xi^i\eta^j = 0$ (this is called the MTW condition).

As shown by Loeper [7], the MTW tensor satisfies the following remarkable identity: if $\xi,\eta \in T_xM$ are two orthogonal unit vectors, then

$$\mathfrak{S}_{(x,x)}(\xi,\eta) = -\frac{3}{2}\frac{\partial^2}{\partial s^2}|_{s=0}\frac{\partial^2}{\partial t^2}|_{t=0}F(t,s) = \mathrm{Sect}_x([\xi,\eta]),$$

where $F(t,s) := \frac{1}{2}d\big(\exp_x(t\xi),\exp_x(s\eta)\big)^2$ and $\mathrm{Sect}_x([\xi,\eta])$ denotes the sectional curvature of the plane generated by $\xi$ and $\eta$. This fact shows that the MTW tensor is a nonlocal version of the sectional curvature and the MTW condition implies nonnegative sectional curvature. In Chapter 12 of the book under review, the author gives a very good introduction to the regularity theory of optimal transport. However, since the book was completed in 2008, it misses some of very recent developments linking the MTW tensor with the geometry of the manifold (see for instance [4] for a recent account on these results).

## 4. Displacement convexity and applications

When $X = Y$ is geodesic space (i.e., a complete separable metric space such that every couple of points can be joined by a minimizing geodesic) and $c(x,y) = d(x,y)^2$, the minimum value in (1) is denoted by $W_2(\mu,\nu)^2$, and $W_2(\mu,\nu)$ is the so-called *Wasserstein distance*. It turns out that the space $P_2(X)$ of probability measures with finite second moment, endowed with the Wasserstein distance, is a geodesic space too. Now, given two measures $\mu_0,\mu_1 \in P_2(X)$, let $(\mu_t)_{t\in[0,1]}$ be a (constant-speed) geodesic joining $\mu_0$ to $\mu_1$. Then the idea is that the behaviour of $\mu_t$ should capture some information on the geometry of the underlying space. More precisely, let us fix a reference measure $\nu$ on $X$, and consider for instance the following functionals on $P_2(X)$:

$$H_N : P_2(X) \to \mathbb{R}, \qquad H_N(\mu) = -\int_X \rho^{1-1/N}\,d\nu,\ \mu = \rho\nu + \mu^s,\ \mu^s \perp \nu,$$

$$H_\infty : P_2(X) \to \mathbb{R}, \qquad H_\infty(\mu) = \begin{cases} \int_X \rho\log(\rho)\,d\nu & \text{if } \mu = \rho\nu, \\ +\infty & \text{otherwise.} \end{cases}$$

When $X = \mathbb{R}^n$, $\nu$ is the Lebesgue measure and $N \geq n$, it was discovered by McCann [10] that the above functionals are convex along Wasserstein geodesics (in short, *displacement convex*). This fact was the starting point for many applications, which we now describe briefly.

• **Gradient flows.** One of the main discoveries of Otto [13] was to understand that many evolution equations can be interpreted as gradient flows in the space $P_2(\mathbb{R}^d)$ of some potential functional with respect to the Wasserstein distance $W_2$. For instance, the gradient flow of $H_\infty$ is the heat equation, while the gradient flow of $H_N$ gives a porous-medium equation. Thanks to the fact that these energy functionals are convex, such an interpretation turns out to be extremely well adapted to proving existence, uniqueness, stability, and asymptotic behavior for solutions.

• **Geometric and functional inequalities.** The displacement convexity is also extremely useful in proving some geometric and functional inequalities. As an example, given two open bounded sets $A, B \subset \mathbb{R}^n$, choose $\mu_0 = \frac{1_A}{|A|}$ and $\mu_1 = \frac{1_B}{|B|}$. Then, the displacement convexity of $H_n$ allows one to prove easily the Brunn-Minkowski inequality:

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}$$

(see [10] for more detail). One great advantage of such a proof is that it relies only on the convexity of $H_n$, and for instance it can be immediately extended to any Riemannian manifold on which $H_n$ is displacement convex.

• **Riemannian manifolds and Ricci curvature bounds.** Understanding whether the energy functionals $H_N$ and $H_\infty$ are displacement convex on a Riemannian manifold when $\nu = \mathrm{vol}$ was an important issue. The combination of results of many authors can be summarized in the following statement:

**Theorem 4.1.** *Let $(X, \nu) = (M, \mathrm{vol})$, and let $N \geq \dim M$. Then $H_N$ (resp. $H_\infty$) is displacement convex if and only if* $\mathrm{Ric} \geq 0$.

This result was fundamental for two reasons: on the one hand, many geometric results on $\mathbb{R}^n$ which one could prove by exploiting the displacement convexity were extended to Riemannian manifolds with nonnegative Ricci curvature. On the other hand, it showed that the convexity of certain energy functionals defined on the space of probability measures allowed rewriting in a more robust way the inequality $\mathrm{Ric} \geq 0$ (and more generally $\mathrm{Ric} \geq K$, $K \in \mathbb{R}$). Thanks to the fact that the displacement convexity was shown to be stable under passage to the limit under the Gromov-Hausdorff convergence of metric spaces [8, 14, 15], this was the starting point for Lott-Villani and Sturm to give a meaning to $\mathrm{Ric} \geq K$ on a metric measured space.

All these results are very well explained in the second and third parts of the book.

## 5. Conclusions

The book under review is written by a leading expert who has made extensive and deep contributions to the subject in the last years. The book is an in-depth, modern, clear exposition of the advanced theory of optimal transport, and it tries to put together in a unified way almost all the recent developments of the theory. Let me recall that the author already wrote an excellent book on the subject few years ago [16], which could be considered as a good starting point before studying the book under review, which treats the subject in much more generality and develops many more different directions. On the other hand, the prerequisites assumed do not go much beyond a first course in analysis, functional analysis, and Riemannian geometry, and the proofs are entirely self-contained. This makes the book accessible to a large audience, including graduate and postgraduate students. Moreover the book is extremely well written and very pleasant to read. In particular, before involved proofs the author first gives a sketch in order to explain the main ideas, and whenever possible he also suggests that the reader skip the proof at a first reading.

Each chapter is followed by notes that provide a short historical background, and some of them are also followed by an appendix which contains some classical results of analysis or geometry which were used in the chapter. Much attention is given to bibliographical and historical notes, and many topics appear in this volume for the first time in book form, e.g., the regularity theory for optimal transport maps or the use of optimal transport to define Ricci curvature bounds on metric spaces.

I strongly recommend this excellent book to every researcher or graduate student in the field of optimal transport. Naturally, it will also be of interest to many mathematicians in different areas, who are simply interested in having an overview of the subject.

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