BOOK REVIEWS

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String topology and cyclic homology, by Ralph L. Cohen, Kathryn Hess, and Alexander A. Voronov, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2006, viii+163 pp., ISBN 3-7643-2182-2, softcover

This book is devoted to some recent developments in our understanding of the topology of free loop spaces of manifolds. The contributions are based on three lecture courses given by the authors at a summer school at Almería, Spain, in the summer of 2003. The two parts treat rather different facets of the theory, and I will try to give some perspective on each of them in turn.

The free loop space of a manifold X is the space $\Lambda X = \mathrm{Map}(S^1, X)$ of maps from S^1 into X. I am deliberately vague on the kind of maps one considers (the minimal requirement being continuity), since that often depends on context and taste. For a classical Riemannian geometer, interest in these spaces comes from the following source. For certain dynamical systems on X, such as the geodesic flow, the closed trajectories can be described as critical points of some (action) functional, and one can use Morse theory and its generalizations to relate existence questions for closed orbits to topological properties of ΛX. An early and still fundamental result in this direction is the theorem of Gromoll and Meyer [20], which asserts the existence of infinitely many geometrically distinct periodic geodesics for any Riemannian metric on simply connected closed manifolds X for which the rational Betti numbers of ΛX are unbounded. Using minimal model techniques, it was shown by Sullivan and Vigué-Poirrier [30] that the latter property holds whenever the rational cohomology ring of X needs more than one generator.

More recently, interest in path and loop spaces has arisen from their importance as basic configuration spaces in string theory. The term *string topology* in the title is a direct reference to this physics lingo. It refers to a whole collection of algebraic structures discovered by Chas and Sullivan [7, 8, 27] on the singular chains of various path spaces of a smooth oriented manifold M, whose underlying operations are a topological model of "string interaction". Indeed, the idea is quite elegant and very easy to explain. Consider a piece of directed string in three-space (the vertical in Figure 1) and a family of pieces of directed string (the horizontals in Figure 1) which move across it. Exactly one of the strings in the family meets the first string, and at this instance one can imagine cutting the two strings at the intersection point and recombining them as shown in Figure 2. This is the essential local operation

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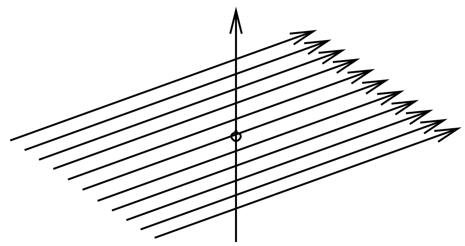

FIGURE 1. A local family of curves meeting another curve transversely

of string topology. The variety of algebraic structures it generates comes from the types of curves considered, and whether one allows the interactions at all points of the string or only at certain points marked in advance. The local picture could also describe a self-intersection, in which case the result of local recombination breaks the curves into two pieces.

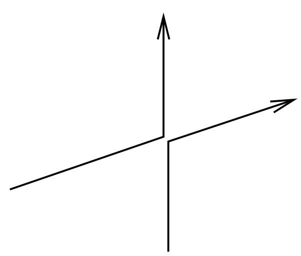

FIGURE 2. Result of local recombination

As already mentioned, in more mathematical terms a "family of strings" is usually a singular chain on a suitable path or loop space on some oriented smooth manifold M. The essential observation of Chas and Sullivan was that if one imposes transversality conditions on the evaluation maps to M, the local operations give rise to transformations from chains to chains. The drawback is that these naive chain level operations are only partially defined. However, standard arguments show that the simplest operations descend to well-defined and well-behaved algebraic structures on homology. The two most prominent examples are

- an associative, graded commutative product of degree $-\dim X$ and a Lie bracket of degree $1-\dim X$ on $H_*(\Lambda X)$, which combine with the operator $\Delta : H_*(\Lambda X) \to H_{*+1}(\Lambda X)$ coming from the circle action rotating the domain to yield a Batalin–Vilkovisky (or BV) algebra structure on $H_*(\Lambda X)$, and
- an involutive Lie bialgebra structure on $H_*^{S^1}(\Lambda X, X)$, the S^1-equivariant homology of the free loop space relative to the subset of constant loops (see [8] for the definitions).

Even though the original paper [7] is still not published, there exist several constructions of the above operations in the literature; see, e.g., [8, 11, 6, 25, 9, 14]. There are further generalizations to operations on chains in spaces of paths with endpoints in suitable smooth submanifolds, where the interactions take place either at the ends of the paths (via intersection theory in the submanifolds) or in the interior of the paths (as described in the geometric picture above). Some of these have been implemented on the homology level in [26, 18], where they are phrased in the language of topological conformal field theories. The chain level version is not yet properly treated in the literature, mainly due to various technical complications. In the naive description above these can be traced back to the fact that the intersection of chains requires transversality. Conceptually, the expected final outcome, however, is quite simple. Depending on the precise context, one obtains a representation of some operad, properad or prop (these are gadgets parametrizing the relevant operations and their compatibilities; see Chapter 2 of Part I of the book or, e.g., [24]) on a category whose morphisms correspond to suitable chains in loop or path spaces in a fixed manifold.

In the recent survey [28], which includes an extensive bibliography, Sullivan describes some of the roots of string topology, as well as its present state of development. In particular, the second part of his survey contains informative accounts of various perspectives on string topology that have emerged so far, and the third part outlines an implementation of the chain level theory corresponding to the involutive Lie bialgebra structure on $H_*^{S^1}(\Lambda X, X)$ mentioned above.

There is a variety of reasons to care about string topology. I will briefly describe my personal interest in the subject, which comes from the close relation to the theory of holomorphic curves with boundary on some Lagrangian submanifold L in a symplectic manifold (M, ω). A holomorphic curve is a map $u : (\Sigma, \partial\Sigma) \to (M, L)$ from some Riemann surface with boundary Σ which satisfies a generalized Cauchy–Riemann equation with respect to an almost complex structure J on M suitably compatible with the symplectic form ω. Fixing the topological type of Σ and a relative homology class $A \in H_2(M, L)$, one obtains a certain moduli space of equivalence classes of such maps representing the homology class A, where two maps are considered equivalent if they are intertwined via an automorphism of the domain. As the equation is elliptic, these moduli spaces have finite (expected) dimension. They are not compact, but can be compactified by adding strata built from earlier (with respect to a suitable partial order) such moduli spaces. With a substantial amount of analytical work, these compactified spaces can be given the structure of smooth weighted branched manifolds with boundary and corners. Such a compactified moduli space can be viewed as a suitable chain of loops (or tuples of loops if Σ had more than one boundary component) by evaluating each point (i.e., holomorphic map u) at the boundary. There is a slight complication here, since each point is only an equivalence class of maps, but this can easily be dealt with. Now the wonderful thing is that the boundary of each such chain can be described in terms of string topology operations applied to other such chains.

One not entirely obvious effect is that in this way one can use information about the string topology of L to obtain restrictions on the possible Langrangian embeddings. This was first noticed by Fukaya, and, applying this circle of ideas with $\Sigma = D^2$, he outlined proofs of several remarkable results in symplectic topology [15]. For example, he completely classified connected, closed, oriented, prime 3-dimensional manifolds which can occur as Lagrangian submanifolds of standard symplectic $\mathbb{C}^3$: the only candidates are products of oriented closed surfaces with the circle, and by a well-known elementary construction these indeed do embed as Lagrangian submanifolds.

There are several ongoing projects (see, e.g., [16, 1, 2, 29, 10]) that aim to make various aspects of the relation of holomorphic curves and string topology precise, and to exploit it in differential topology, symplectic topology, and Hamiltonian dynamics. It appears that in many of these applications, some chain level discussion is essential, but there is some freedom in "shifting the analytic difficulties" between the holomophic curve side and the topological side.

In the first part of the book under review, Cohen and Voronov take the reader on a guided tour through some of the tools used in string topology, with an emphasis on homotopy-theoretic methods. Chapter 1 starts with a review of intersection theory in manifolds, and introduces the loop product, loop bracket, BV-operator Δ, and string bracket mentioned above on the homology level. It also presents the Cohen–Jones approach [11] to the loop product via stable homotopy theory, and it briefly sketches one relation of string topology to Hochschild cohomology (I will return to this point below). Chapter 2 gives a rapid review of operads and props, as well as algebras over them, with special attention to the cacti operad introduced by Voronov [31], which can be used to describe part of string topology. Chapter 3 discusses the topological field theory perspective on string topology mentioned above, and Chapter 4 is dedicated to Floer homological interpretations on the cotangent bundle. This is one instance where the holomorphic curve picture described above has recently been worked out in some detail [1, 2]. The last chapter of Part I describes some ongoing work of the second author on higher-dimensional generalizations of string topology, with general spheres replacing the loops or strings in the constructions. This is another aspect of the theory that is still in its infancy, with promising lines for further research.

As often happens with guided tours—I presume most tourists will agree—the customer interested in details will want to consult additional sources, but at the very least he or she will have a good enough overview of the basic ideas to make an informed choice of what to read next.

The second part of the book is concerned with topological cyclic homology, and more specifically it describes the construction of algebraic models amenable to computations. As I am far from an expert on this subject, I will restrict myself to some background remarks. In their preparation, I found the surveys by Berrick [3] and Madsen [23] and the monograph of Loday [22] very helpful.

The cyclic (co)homology of an associative algebra first appeared in the works of Connes, Loday and Quillen, and Tsygan in the early 1980s. To describe the idea, I start by briefly recalling the definition of Hochschild homology. Suppose that A is a unital associative algebra over some ring k, say $k = \mathbb{Q}$ for simplicity. Given an A-bimodule M, one considers the sequence $C_\bullet(A, M)$ of modules $C_n(A, M) = M \otimes A^{\otimes n}$. Together with the structure maps

$$d_i(m, a_1, \ldots, a_n) = \begin{cases} (ma_1, a_2, \ldots, a_n), & i = 0, \\ (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n), & 0 < i < n, \\ (a_n m, a_1, \ldots, a_{n-1}), & i = n, \end{cases} \tag{1}$$

$$s_i(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_i, 1, a_{i+1}, \ldots, a_n), \quad 0 \le i \le n, \tag{2}$$

the sequence is what is known as a simplicial module. It is sometimes called the cyclic bar construction of A with coefficients in M. By definition, the Hochschild homology of A with coefficients in M is

$$HH_*(A,M) := H_*(C_\bullet(A,M),d), \quad \text{where } d = \sum_i (-1)^i d_i.$$

In the special case that $M = A$, one simply writes $HH_*(A)$. An important property of Hochschild homology is its Morita invariance,

$$HH_*(\mathcal{M}_r(A)) \cong HH_*(A),$$

where $\mathcal{M}_r(A)$ denotes the $r \times r$ matrices with coefficients in A. This isomorphism is induced from a trace map $\mathrm{tr} : C_n(\mathcal{M}_r(A), \mathcal{M}_r(A)) \to C_n(A,A)$ given as

$$\mathrm{tr}(M_0 \otimes \cdots \otimes M_n) := \sum_{0 \le i_0,\ldots,i_n \le r} (M_0)_{i_0 i_1} \otimes (M_1)_{i_1 i_2} \otimes \cdots \otimes (M_n)_{i_n i_0},$$

which is easily checked to be a simplicial map. This trace map plays an important role in the definition of the Dennis trace map

$$\mathrm{Tr} : H_n(GL(A),k) \to HH_n(A), \tag{3}$$

which is constructed from the composition

$$H_n(GL_r(A),k) \xrightarrow{\iota_*} HH_*(k[GL_r(A)]) \xrightarrow{f} HH_n(\mathcal{M}_r(A)) \xrightarrow{\mathrm{tr}} HH_n(A). \tag{4}$$

Here $GL_r(A) \subset \mathcal{M}_r(A)$ denotes the multiplicative subgroup of invertible matrices. To explain this sequence of maps, note that in analogy with the cyclic bar construction for algebras above, we can form the cyclic bar construction $B_\bullet^{\mathrm{cyc}}(G,Y)$ of a group G with coefficients in a two-sided G-space Y, with $B_n(G,Y) = Y \times G^n$, using exactly the same formulas (1) and (2) as before. One gets a simplicial set and, by definition, the homology of a group G is computed by using the trivial two-sided G-space $Y = \{*\}$. In this case $B_\bullet G := B_\bullet^{\mathrm{cyc}}(G,*)$ is simply called the bar construction of G. Now the map ι_* in (4) is induced from the map $\iota : B_\bullet GL_r(A) \to C_\bullet(k[GL_r(A)])$ defined as $\iota(*, g_1, \ldots, g_n) = ((g_1 \cdots g_n)^{-1}, g_1, \ldots, g_n)$, and the map f is induced from the "fusion map", which maps a formal linear combination of invertible matrices to an actual sum. Note that both of these maps are simplicial. The lucky fact is that, while the fusion map does not stabilize well as r increases (for $GL_*(A)$ one adds a diagonal 1, for $\mathcal{M}_*(A)$ one adds a row and column of zeroes), the composition does, giving rise to a map (3) as claimed.

A further simple but important observation is that in all of the above examples, there is an additional cyclic symmetry, in the sense that the structure maps d_i and s_i are compatible with an action of the cyclic group $\mathbb{Z}_{n+1}$ on C_n. To be explicit, on C_n we have the following relations:

$$d_i t = t d_{i-1} \text{ for } 1 \le i \le n \text{ and } d_0 t = d_n,$$
$$s_i t = t s_{i-1} \text{ for } 1 \le i \le n \text{ and } s_0 t = t^2 s_n.$$

A simplicial object with this additional structure is called a cyclic object, a fundamental notion introduced by Connes [12]. In the case of the Hochschild complex, the action is generated by $t(a_0, \ldots, a_n) = (a_n, a_0, \ldots, a_{n-1})$, and for the bar construction $B_\bullet G$ of a group it is generated by $t(*, g_1, \ldots, g_n) = (*, (g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1})$.

Notice that the above relations imply that the Hochschild boundary map d descends to a well-defined map b on the quotient of $C_n(A,A)$ by the action of $(1-t)$. Now the cyclic homology $HC_*(A)$ of the algebra A is simply defined as the homology of this quotient complex [13], at least under our assumption that $k = \mathbb{Q}$ (see, e.g., [22] for the general construction using suitable bicomplexes). Connes motivation in [13] was to generalize the classical Chern character from differential geometry,

$$\mathrm{ch} : K_0(X) \to H^{ev}_{deRham}(X).$$

which associates to each vector bundle its total Chern class. Here generalization on the one hand meant extending it to more general algebras than the commutative algebra of functions on a manifold, in particular noncommutative ones. But there is also an extension to higher K-theory, which is inspired by the Dennis trace map. In fact, notice that all the maps in (4) are in fact maps of cyclic objects, so they induce maps on cyclic homology. Moreover, the projection map from $B_\bullet G$ to the quotient by $(1-t)$ give rise to a map from the group homology to the cyclic homology of the group, so analogously to the Dennis trace map we obtain the Chern character

$$\mathrm{ch} : H_*(GL(A); k) \to HC_*(A).$$

In fact it is customary to view it as a map from the K-theory of A to a variant of cyclic homology called negative cyclic homology, but I will not pursue this point here (again, see, e.g., [22] for details).

All this algebra leaves us with an obvious question: What does all of this have to do with the free loop space? In the algebraic discussion up to this point, we have only used the face operators d_i (and the cyclic operator t). However, the simplicial structure allows us to form geometric realizations of all our objects. It was noticed early on that the geometric realization $|X_\bullet|$ of a cyclic set or space $X_\bullet$ has a natural S^1-action; see, e.g., [5, 19, 21]. Given a topological group G, we can view $Y = G$ as a two-sided G-space and consider the cyclic bar construction $B^{\mathrm{cyc}}_\bullet G := B^{\mathrm{cyc}}_\bullet(G, G)$ of G, which by our above discussion is a cyclic space. Now we have a projection map $p : B^{\mathrm{cyc}}_\bullet G \to B_\bullet G$, defined as $p(g_0, \dots, g_n) = (*, g_1, \dots, g_n)$, which is a map of simplicial spaces. Passing to geometric realizations, and using the natural S^1-action on $|B^{\mathrm{cyc}}_\bullet G|$ we get a continuous map

$$S^1 \times |B^{\mathrm{cyc}}_\bullet G| \to |B^{\mathrm{cyc}}_\bullet G| \overset{p}{\to} |B_\bullet G| \overset{\simeq}{\to} BG,$$

and it turns out that the adjoint $|B^{\mathrm{cyc}}_\bullet G| \to \mathrm{Map}(S^1, BG)$ of this map is a homotopy equivalence. This construction in fact extends to more general objects G than just groups. In particular one can use the Moore loop space ΩX of a pointed topological space X in place of G. In this way one obtains a homotopy equivalence

$$|B^{\mathrm{cyc}}_\bullet \Omega X| \overset{\simeq}{\to} \mathrm{Map}(S^1, B\Omega X) \simeq \Lambda X, \tag{5}$$

where in the last step we have used the fact that $B\Omega X \simeq X$. Arguing along these lines, one obtains canonical isomorphisms [19]

$$HH_*(S_*(\Omega X)) \equiv H_*(\Lambda X)$$

and

$$HC_*(S_*(\Omega X)) \equiv H^{S^1}_*(\Lambda X),$$

where $S_*(\Omega X)$ denotes the strictly associative algebra of singular chains on the Moore loop space with the Pontryagin product.

At this point I should mention that there are other Hochschild and cyclic (co)homology constructions that yield the homology of the free loop space. In fact, for simply connected spaces X, one has [21] (see also the end of the first chapter of Part I of the book)

$$HH^*(S^*(X), S^*(X)) \cong H_{*+\dim X}(\Lambda X), \tag{6}$$

which also admits a cyclic version. Here $S^*(X)$ denotes singular cochains, with algebra structure given by cup product. The importance of this construction is that Hochschild *co*homology of any algebra has the structure of a Gerstenhaber algebra [17]. In fact, for cochains on a smooth closed oriented manifold, Félix and Thomas [14] proved that, with coefficients in a field of characteristic zero, the Gerstenhaber algebra structure lifts to a BV algebra structure, and (6) is an isomorphism of BV algebras, where on the right one considers the string topology BV-structure mentioned in the first part of this review.

Topological Hochschild and cyclic homologies were defined with the goal of extending the tools from algebraic K-theory for rings to "rings up to homotopy". According to [23], Goodwillie first suggested replacing the algebra A by an Eilenberg–MacLane spectrum and the tensor product by the smash product of spectra in the above algebraic constructions. For the Hochschild story, this was carried out by Bökstedt, and the cyclic version was developed by Bökstedt, Hsiang and Madsen [4]. There they defined topological cyclic homology, which assigns to each topological space X and prime p a topological spectrum $TC(X;p)$. Moreover, they generalized the Chern character to the cyclotomic trace, which maps from K-theory to topological cyclic homology, and used it to prove a K-theoretic version of the Novikov conjecture.

So far, very few explicit computations for topological cyclic homology are known, and the basic aim of Part II of the book is to construct algebraic models for the topological cyclic homology of a space X which are amenable to such computations. Here *model* means a cochain complex (or algebra) such that, after tensoring with the finite field $\mathbb{F}_p$, its cohomology is isomorphic to the mod p spectrum cohomology of $TC(X;p)$. After collecting some preliminaries in Chapter 1, Chapter 2 describes the construction of certain algebraic models for the free loop space. Chapter 3 explains the construction of a model for the homotopy orbit space for the S^1-action on the loop space. Finally, in Chapter 4 the pth power map (for $p=2$) is modelled, and the previous machinery is put together to construct the advertised model of the mod 2 topological cyclic homology.

To summarize, this book gives an introduction to some exciting recent developments in the topology of free loop spaces. It should be a good starting point for a more detailed study of the rapidly developing literature on this subject, and given the pace of these advances, this maybe the best that could be hoped for. In fact, it is not a very daring prediction that eventually our collective understanding of the topology of free loop spaces will allow us to combine the two aspects of the theory which are presented somewhat disjointly in this book.

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