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Multiplicative invariant theory, by Martin Lorenz, Encyclopaedia of Mathematical Sciences, vol. 135, Invariant Theory and Algebraic Transformation Groups, VI, Springer-Verlag, Berlin, 2005, xii+177 pp., US$109.00, ISBN 3-540-24323-2

Let $\mathbb{K}$ be a commutative integral domain and let $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ denote the polynomial ring over $\mathbb{K}$ in the n variables $x_1, x_2, \ldots, x_n$. If H is a group of automorphisms of the free $\mathbb{K}$-module $V = \mathbb{K}x_1 + \mathbb{K}x_2 + \cdots + \mathbb{K}x_n$, that is, if H is a subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{K})$, then the linear action of H on V extends uniquely to a $\mathbb{K}$-algebra action on S. The relationship between S, H, the H-stable prime ideals of S, and the subring of H-invariants $S^H = \{s \in S \mid s^h = s \text{ for all } h \in H\}$ is the realm of (additive) invariant theory. The adjective "additive" is by no means a standard part of the name, but it is useful in the context of this review. Note that the full group of $\mathbb{K}$-automorphisms of S is appreciably larger than $\mathrm{GL}_n(\mathbb{K})$, since for example it contains maps of the form $x_i \mapsto x_i$ for all $i \neq 1$ and $x_1 \mapsto x_1 + f(x_2, \ldots, x_n)$, but, for the most part, additive invariant theory is concerned with $\mathrm{GL}_n(\mathbb{K})$ and its subgroups.

Now suppose we modify S slightly by adjoining the inverses $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$. Then the new ring R is usually written as $R = \mathbb{K}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$, and clearly few subgroups of $\mathrm{GL}_n(\mathbb{K})$ will actually determine $\mathbb{K}$-automorphism groups of R. Indeed, if $X = \langle x_1, x_2, \ldots, x_n\rangle \cong \langle x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_n\rangle$ is the multiplicative subgroup of the unit group of R, then R is isomorphic to $\mathbb{K}[X]$, the group ring of X over $\mathbb{K}$, and the unit group $R^\bullet$ of R is easily seen to be given by $R^\bullet = \mathbb{K}^\bullet \times X$. In particular, any $\mathbb{K}$-automorphism of R must stabilize $R^\bullet$ and act faithfully as automorphisms on this group. Note that X is isomorphic to the additive lattice $\mathbb{Z}^n$, so $\mathrm{Aut}(X) \cong \mathrm{GL}_n(\mathbb{Z})$, and if G is any subgroup of $\mathrm{Aut}(X)$, then the action of G on X extends uniquely to a $\mathbb{K}$-algebra action on R. The relationship between R, G, the G-stable prime ideals of R, and the fixed ring R^G is the stuff of multiplicative invariant theory.

The two invariant theories are obviously similar, and yet they are different in many ways. Additive invariant theory is classical, well over a hundred years old, while the multiplicative version has a much shorter pedigree. Admittedly there are earlier results to be found in the study of algebraic groups, field theory, Lie algebras and group algebras, as we will see below, but multiplicative invariant theory itself was literally born and named in the three fundamental papers [Fa1, Fa2, Fa3] by D. R. Farkas in the mid 1980's. Thus, the book under review discusses the state of this subject at a mature, but still young, twenty years of age.

An interesting example in classical invariant theory is as follows. Suppose we are given $n = m^2$ indeterminates x_{ij} with $1 \leq i, j \leq m$. We can think of these elements as the entries of an $m \times m$ generic matrix $[x_{ij}]$, and we can let $H = \mathrm{SL}_m(\mathbb{K})$ act linearly on these variables by $h\colon [x_{ij}] \mapsto [x_{ij}]h$ for any $h \in H$, where the latter product is $m \times m$ matrix multiplication. If $\mathbb{K}$ is an infinite field, then H is an infinite group that has no nontrivial finite homomorphic images and has no nontrivial fixed

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points on $V = \sum_{i,j} \mathbb{K}x_{ij}$. Nevertheless, $\det[x_{ij}] \in \mathbb{K}[x_{ij} \mid \text{all } i, j] = S$ is a nontrivial H-fixed element in the ring.

On the other hand, this phenomenon cannot happen in the multiplicative case. If $G \subseteq \mathrm{GL}_n(\mathbb{Z})$ acts on $R = \mathbb{K}[X]$, then G permutes the basis X, and consequently the fixed ring R^G has as a $\mathbb{K}$-basis the sums of the finite G-orbits on X. In particular, $R^G \subseteq \mathbb{K}[D]$, where D is the subgroup of X consisting of all elements having finitely many G-conjugates. But $X \cong \mathbb{Z}^n$, so $D \cong \mathbb{Z}^m$, for some $m \leq n$. Thus D is finitely generated, and therefore G acts as a finite group $\widetilde{G}$ on D with $R^G = \mathbb{K}[D]^{\widetilde{G}}$. This, of course, reduces many questions in multiplicative invariant theory to the action of finite groups. For example, it follows from the above that the fixed ring R^G is necessarily a finitely generated $\mathbb{K}$-algebra.

There is, in fact, a decidedly nontrivial version of the above observation, at least when $\mathbb{K}$ is a field. Namely, suppose P is a G-stable prime ideal of R and assume, almost without loss of generality, that the map from X to its image $\overline{X} \subseteq \overline{R} = R/P$ is one-to-one. Then $\overline{R} = \mathbb{K}\overline{X}$ is a domain spanned over $\mathbb{K}$ by the group of units $\overline{X} \cong X$ and, of course, G acts on $\overline{R}$ and stabilizes $\overline{X}$. Note that D is a pure subgroup of X, so $X = D \times Y$ for some subgroup Y, and then a result of J. E. Roseblade [Ro], based on the keen insight of G. M. Bergman [Be], asserts that $\overline{R} = (\mathbb{K}\overline{D})[\overline{Y}]$ is the group ring of $\overline{Y} \cong Y$ over the subring $\mathbb{K}\overline{D}$. As a consequence, Farkas noted that the G-fixed points in the field of fractions of $\overline{R}$ are all contained in the field of fractions of $\mathbb{K}\overline{D}$, where again G acts like a finite group.

Let us consider another example. Again let $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring, this time over a field $\mathbb{K}$, and let the symmetric group $H = \mathrm{Sym}_n$ act on S by permuting the variables. Then S^H is the ring of symmetric polynomials over $\mathbb{K}$ and, as is well known, this is the polynomial ring in the elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_n$. In other words, S^H is also a polynomial ring. Indeed, the celebrated result of G. C. Shephard, J. A. Todd and C. Chevalley [ST, Bo] asserts that if $\mathbb{K}$ is any field and if H is a finite subgroup of $\mathrm{GL}_n(\mathbb{K})$ with $|H| \neq 0$ in $\mathbb{K}$, then S^H is a polynomial ring if and only if H is generated by pseudo-reflections, that is, by matrices of finite order fixing all the points in some hyperplane. To see how S^H might fail to be a polynomial ring in general, just let $\mathbb{K}$ have characteristic different from 2, let $n \geq 2$ and take H to be the group of order 2 whose nonidentity element sends each x_i to its negative. Then S^H is spanned by all monomials in $x_1, x_2, \ldots, x_n$ of even degree and, from the equation $(x_1x_2)(x_1x_2) = (x_1x_1)(x_2x_2)$, it follows that unique factorization does not hold in the fixed ring.

The natural analog in multiplicative invariant theory is to consider when $R^G = \mathbb{K}[X]^G$ is a group ring over $\mathbb{K}$. But this almost never occurs. Indeed, using the fact that R^G has a $\mathbb{K}$-basis consisting of finite G-orbit sums and using the fact that $R^\bullet = \mathbb{K}^\bullet \times X$ is so meager, it follows easily that R^G is a group ring if and only if G acts trivially on D. In other words, in our previous notation, this occurs precisely when $\widetilde{G} = 1$, so that $R^G = \mathbb{K}[D]^{\widetilde{G}} = \mathbb{K}[D]$. On the other hand, suppose $R = \mathbb{K}[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1}]$ and let G be the diagonal subgroup of $\mathrm{GL}_n(\mathbb{Z})$, so that G is an elementary abelian group of order 2^n whose elements map x_i to $x_i^{\pm}$ for all i. Then it is not hard to see that R^G is the ordinary polynomial ring $\mathbb{K}[y_1, y_2, \ldots, y_n]$ with $y_i = x_i + x_i^{-1}$. Thus, the more appropriate question here is to determine when R^G is a polynomial ring or more generally a monoid algebra. Note that if $n \geq 2$ and if $g \in G$ maps each x_i to its inverse, then it is known that the fixed ring $R^{\langle g \rangle}$ is not even a monoid algebra.

An old result of N. Bourbaki [Bo] constructs interesting examples in a rather beautiful manner based on classical Lie algebra theory. Specifically, let $X = \Lambda(\Phi)$ be the weight lattice of a reduced root system Φ and let G be the Weyl group of Φ. Then the fixed ring $\mathbb{Z}[X]^G$ is a polynomial algebra over $\mathbb{Z}$ with the G-orbit sums of a set of fundamental weights being algebraically independent generators. This fact actually holds over all domains $\mathbb{K}$ and the converse, due to Farkas [Fa1] for $\mathbb{K} = \mathbb{C}$ and implicit in the work of R. Steinberg [St], is true for most ground rings. Indeed, let G be a finite group of automorphisms of X and let $\mathbb{K}$ be a regular commutative ring with $|G| \neq 0$ in $\mathbb{K}$. If $\mathbb{K}[X]^G$ is a polynomial algebra over $\mathbb{K}$, then X is isomorphic to the weight lattice of some reduced root system with G acting as the Weyl group.

A surprising difference between the two theories shows up in the structure of the Picard group of the fixed ring. Recall that if T is a commutative ring, then a T-module P is said to be invertible if $P \otimes_T Q \cong T$ for some T-module Q, and then $\mathrm{Pic}(T)$ is the set of isomorphism classes of invertible T-modules with tensor product multiplication. If $\mathbb{K}$ is a field, then it is known that the embeddings $\mathbb{K} \to \mathbb{K}[X] = R$ and $\mathbb{K} \to \mathbb{K}[x_1, x_2, \ldots, x_n] = S$ determine isomorphisms $\mathrm{Pic}(R) \cong \mathrm{Pic}(\mathbb{K}) \cong \mathrm{Pic}(S)$, and of course $\mathrm{Pic}(\mathbb{K}) = 0$. Furthermore, if $H \subseteq \mathrm{GL}_n(\mathbb{K})$, then M-C. Kang [Ka] has shown that the embedding $S^H \to S$ yields an isomorphism $\mathrm{Pic}(S^H) \cong \mathrm{Pic}(S) = 0$. In other words, the Picard group of S^H is always trivial. On the other hand, if G is a finite subgroup of $\mathrm{GL}_n(\mathbb{Z})$, then there is a cohomological formula that describes $\mathrm{Pic}(R^G)$, and computations show that this group can be nontrivial.

Of course, the field of fractions of $R = \mathbb{K}[X]$ and of $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ are the same. Thus, when it comes to studying rational function fields, the multiplicative and additive invariant theories merge to some extent. There are two basic questions here, the first being Noether's problem. Let F/K be a rational extension of fields, so that $F = K(t_1, t_2, \ldots, t_n)$ with algebraically independent generators $t_1, t_2, \ldots, t_n$. If G is a finite group of K-automorphisms of F, then Noether's problem asks whether the fixed field F^G is also rational over K. The answer here is generally negative. An interesting special case occurs when the finite group G regularly permutes the generators of F/K. Specifically, let $\mathbb{K}$ be a field and let $\mathbb{K}(x_g \mid g \in G)$ be the rational extension of $\mathbb{K}$ having one generator x_g for each $g \in G$. The natural action of G is given by $(x_g)^h = x_{gh}$, and we let $\mathbb{K}(G)$ denote the fixed field. In [Ln], H. W. Lenstra Jr. determined precisely when $\mathbb{K}(G)/\mathbb{K}$ is rational for any abelian group G. For example, if $\mathbb{K} = \mathbb{Q}$, then $\mathbb{K}(G)/\mathbb{K}$ is not rational for $G = C_8$, the cyclic group of order 8, and for infinitely many cyclic groups C_p of prime order.

In view of these counterexamples, it is more appropriate to ask whether $E = F^G$ necessarily satisfies some weaker version of rationality over K. For example, E/K is said to be stably rational if there exists a rational extension E' of E that is also rational over K, and E/K is said to be retract rational if E is the field of fractions of some K-algebra T that is a retract of a localized polynomial ring. Specifically, the latter means that T is a subring of some $T' = K[x_1, x_2, \ldots, x_n][1/s]$ and there is a homomorphism $\pi\colon T' \to T$ that is the identity on T. While the above mentioned counterexamples are not even stably rational, D. J. Saltman [Sa2] has shown that $\mathbb{Q}(C_p)/\mathbb{Q}$ is retract rational for all primes p.

We remark that Noether's problem was originally motivated by considerations in constructive Galois theory, the inverse Galois problem. Indeed, when $\mathbb{K}(G)/\mathbb{K}$ is

rational, then G must be a Galois group over $\mathbb{K}$. Furthermore, it was shown by Saltman [Sa1] for $\mathbb{K}$ infinite and by F. DeMeyer and T. McKenzie [DM] in general that even if $\mathbb{K}(G)/\mathbb{K}$ is merely retract rational, then there exists a generic polynomial with Galois group G. In other words, there exists a rational field $\mathbb{K}(t_1, t_2, \ldots, t_m)$ and a separable polynomial $f(t_1, t_2, \ldots, t_m)(x) \in \mathbb{K}(t_1, t_2, \ldots, t_m)[x]$ with Galois group G that has the following universal property. If E/F is any Galois extension with Galois group G and with $F \supseteq \mathbb{K}$, then there exist $a_1, a_2, \ldots, a_m \in F$ so that E is the splitting field over F of the separable polynomial $f(a_1, a_2, \ldots, a_m)(x) \in F[x]$.

One can argue that the real relationship of this problem to multiplicative invariant theory occurs when $F = K(X)$ is the field of fractions of $K[X]$ and when G acts multiplicatively on $X \cong \mathbb{Z}^n$. In this case, Farkas [Fa3] has shown that if $G \subseteq \mathrm{GL}_n(\mathbb{Z})$ is a reflection group, that is, if G is generated by pseudo-reflections of order 2, then $K(X)^G$ is rational over K. More recently, N. Lemire [Lm] proved that if Φ is a reduced root system, $G = \mathrm{Aut}(\Phi)$, and if X is rationally isomorphic to $\Lambda(\Phi)$, then again $K(X)^G$ is rational over K.

The second basic problem is intimately related to the structure of matrix rings. Let $\mathbb{K}$ be an algebraically closed field, write M_n for the ring $\mathrm{M}_n(\mathbb{K})$ of $n \times n$ matrices over $\mathbb{K}$, and let $r \geq 2$ be an integer. Then $G = \mathrm{PGL}_n(\mathbb{K})$ acts on M_n^r, the space of r-tuples of M_n, by simultaneous conjugation, so that $(A_1, A_2, \ldots, A_r)^g = (A_1^g, A_2^g, \ldots, A_r^g)$, where $A_\ell^g = g^{-1} A_\ell g$. It follows that G also acts on $\mathcal{O}(M_n^r)$, the ring of polynomial functions on M_n^r. Specifically, the latter ring is generated over $\mathbb{K}$ by the variables $x_{ij}^{(\ell)}$, where $x_{ij}^{(\ell)}(A_1, A_2, \ldots, A_r)$ is the (i,j)th entry of the matrix A_ℓ. If $\mathcal{K}(M_n^r)$ is the field of fractions of $\mathcal{O}(M_n^r)$, then the goal is to study the fixed subfield $\mathcal{K}(M_n^r)^G$. As was shown by C. Procesi [Pr2], this fixed subfield is actually isomorphic to the center of the generic division algebra $\mathrm{UD}(\mathbb{K}, n, r)$, and hence it is of great interest in noncommutative ring theory. Furthermore, multiplicative invariant theory comes into play here since a result of E. Formanek [Fo1] and Procesi [Pr1] asserts that $\mathcal{K}(M_n^r)^{\mathrm{PGL}_n(\mathbb{K})} \cong \mathbb{K}(L_{n,r})^{\mathrm{Sym}_n}$ for some lattice $L_{n,r}$ over the symmetric group Sym_n. The rationality of the extension $\mathbb{K}(L_{n,r})^{\mathrm{Sym}_n}/\mathbb{K}$ is a particularly intractable problem. It is known to be true for $n = 2$ by the work of J. J. Sylvester [Sy], and for $n = 3$ and 4 by the work of Formanek [Fo1, Fo2]. Stable rationality for $n = 5$ and 7 is due to C. Bessenrodt and L. Le Bruyn [BL]. Retract rationality of the extension when n is prime is due to Saltman [Sa2].

Finally, the polynomial ring $S = \mathbb{K}[x_1, x_2, \ldots, x_n]$ is graded by total degree, with each component being a finitely generated free $\mathbb{K}$-module. Furthermore, if $H \subseteq \mathrm{GL}_n(\mathbb{K})$, then H preserves the grading. Thus H acts in a locally finite manner on S, and S^H is a graded subring. In particular, additive invariant theory can use the surprisingly powerful techniques of graded ring theory to good advantage. On the other hand, if $G \subseteq \mathrm{GL}_n(\mathbb{Z})$, then G does not preserve any natural grading on $R = \mathbb{K}[X]$ and, if G is infinite, then it does not act in a locally finite manner.

In spite of this, multiplicative invariant theory does have its own unique features and special tricks. For one thing, there is little dependence on the base ring here. Indeed, $R = \mathbb{K} \otimes_{\mathbb{Z}} \mathbb{Z}[X]$ and the nature of the basis for the fixed ring implies that $R^G = \mathbb{K} \otimes_{\mathbb{Z}} \mathbb{Z}[X]^G$. Thus properties of the fixed ring, like being finitely generated, transfer immediately from the integer case to the more general situation. Furthermore, when studying the fixed ring, it suffices to assume that G is finite, and a classical result of C. Jordan [Jo] asserts that, for fixed n, there are only finitely many conjugacy classes of finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$. Thus it is theoretically

possible, using computer algebra applications like GAP, to tabulate all relevant information for small values of n. For example, in the book being reviewed, the Picard groups are computed for all $n \leq 3$ and the fixed rings are described for all $n \leq 2$, indicating which of these are monoid algebras.

Multiplicative Invariant Theory by Martin Lorenz is a beautiful book on an exciting new subject, written by an expert and major contributor to the field. Indeed, Chapter 4 on class groups is substantially due to the author [Lo1], as is much of the progress discussed in Chapter 8 on understanding when the fixed ring R^G inherits the Cohen-Macaulay property [Lo3]. Chapter 5 on Picard groups benefits greatly from his insight [Lo2]. The book includes all of the above discussed material and a good deal more. Most of the proofs have been completely reworked, and many of the results appear to be new. The author is especially careful to explain where each chapter is going, why it matters, and what background material is required. The last chapter on open problems, with a good deal of annotation, is certainly welcome, since there is much yet to be done. Be aware, this is definitely a research monograph. The subject matter is broad and deep, and the prerequisites on the reader can sometimes be daunting. Still, it is wonderful stuff and well worth the effort. I found almost no typos and just a few printing errors apparently due to formatting changes. The author has a web page, `http://www.math.temple.edu/~lorenz/MITcorrections.pdf`, where some corrections are listed.

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