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Random walks on infinite graphs and groups, by Wolfgang Woess, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, 2000, xi+334 pp., \$64.95, ISBN 0-521-55292-3

A *random walk*, a synonym in a loose sense for a time homogeneous Markov chain on a countable state space, used to be an object that probabilists were handling as a typical stochastic process. To be more precise, given a countable set V on which a walker moves around, we consider a probability law under which the walker transfers from one site to another. This law is described by a non-negative valued function p on $V \times V$ such that $\sum_{y\in V} p(x,y) = 1$. We consider $p(x,y)$ as the probability that a walker at the site x moves to y in one step. The n-step transition probability $p_n(x,y)$ is then defined by

$$\sum_{x_1,\dots,x_{n-1}\in V} p(x,x_1)p(x_1,x_2)\cdots p(x_{n-1},y)$$

which is thought of as the probability that a walker starting from the site x is found at y after n-step random movements. As a stochastic process, a random walk is a Markov process in the sense that the future of a random walker depends only on the present site and not on the past record.

What probabilists, including the author of this book, are interested in is the behavior of $p_n(x,y)$ as n goes to infinity. For instance, one queston is to determine whether the series

$$g(x,y) = \sum_{n=0}^{\infty} p_n(x,y) \tag{1}$$

converges or not. Another is to find out what the asymptotic of $p_n(x,y)$ itself is.

The first question is the so-called "recurrence-transience" problem. Recall that, if (1) diverges, then a walker tends to come back again and again in finite regions in probability one. Whereas, if (1) converges, then the walker goes to "infinity" in probability one. Historically this question originates in Polya's observation in 1921 on the simple random walk on the hyper-cubic lattice $\mathbf{Z}^k$. What Polya showed is that the simple random walk on $\mathbf{Z}^2$ is recurrent, while it is transient on $\mathbf{Z}^k$ with $k \geq 3$.

The second question is the "(local) limit" problems which include, among others, the question whether the central limit theorem and/or a large deviation property holds. Needless to say, the prototype of these limit problems is the Gauss-Laplace theorem on the convergence of the Bernoulli sequence to the normal distribution.

A more sophisticated problem is to describe, in the transient case, the "ideal boundaries" (e.g., the Martin boundary or the Poisson boundary) which consist of "points" at infinity reached by the random walker when n goes to infinity. The ideal boundaries are related to the problem on the existence of positive or bounded harmonic functions.

In short, the object is simple to deal with, and the questions are easy to understand. Traditionally, potential theory (together with Fourier analysis in a special

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case) provides us sufficient terminologies and methods to develop the "general" theory of random walks (see J. L. Doob [3] and F. Spitzer [7] for these classical methods). If we wish to go further, however, the matter is not so hands-down. It turns out that geometric considerations are required to establish many fruitful results on random walks. What is then the geometry in this case ? It is the geometry of graphs. Here a graph structure on V is introduced in such a way that two sites x, y are adjacent if and only if $p(x, y) > 0$.

We can start the theory of random walk the other way around, which is actually the standpoint of this book as the author states in the preface that "*what we have in mind is to start with a graph, groups, etc. and investigate the interplay between the behaviour of random walks on those objects on one hand and properties of the underlying structure itself on the other.*" For instance, in this view, the theory of discrete groups is naturally connected to the theory of random walks. Namely we relate properties of random walks on the Cayley graph associated with a finitely generated group G to the group structure of G, say group growth, amenability, hyperbolicity, etc. (see H. Kesten [4] as a pioneer work). If the reader is *geometry-oriented*, he would agree with the opinion that this standpoint serves as a natural "guiding principle" to develop the "fine" theory of random walk. In fact, we may undertake, if not always, a parallel discussion to the analysis of Laplace operators on (open) Riemannian manifolds (see [5] and N. Th. Varopoulos, L. Saloff-Coaste and T. Coulhon [8] for instance.)

Let me give a brief account of the "geometric" view of random walks. Consider a connected graph $X = (V, E)$, V being the set of vertices and E being the set of all oriented edges. Let $p : E \longrightarrow \mathbf{R}$ be a function satisfying $p(e) \geq 0$ $(e \in E)$ and $\sum_{e \in E_x} p(e) = 1$ where E_x denotes the set of edges whose origin $o(e)$ is x (we also denote by $t(e)$ and $\overline{e}$ the terminus and the inverse edge of e, respectively). If there exists a positive valued function m on V such that $p(e)m(o(e)) = p(\overline{e})m(t(e))$, the random walk associated with p is said to be *symmetric* (or *reversible*). The simple random walk is the one given by $p(e) = (\deg\, o(e))^{-1}$ where $\deg x = \# E_x$. Associated with p, we define the transition operator P acting on functions on V by

$$(Pf)(x) = \sum_{e \in E_x} p(e) f(t(e)).$$

Note that the n-step transition probability $p_n(x, y)$ is the kernel function of P^n in the sense that $(P^n f)(x) = \sum_{y \in V} p_n(x, y) f(y)$, and that if we put $f_n = P^n f$, then

$$f_{n+1} - f_n = (P - I) f_n, \quad f_0 = f$$

which may be regarded as a discrete analogue of the heat equation

$$\frac{\partial f}{\partial t} = \Delta f.$$

The operator $P - I$ looks much more like the Laplace operator if we introduce the space of "1-forms" $C_-(E)$ and the operator $d : C(V) \longrightarrow C_-(E)$, $\delta : C_-(E) \longrightarrow C(V)$ by

$$C_-(E) = \{\omega : E \longrightarrow \mathbf{R};\ \omega(\overline{e}) = -\omega(e)\}$$

$$df(e) = f(t(e)) - f(o(e)), \quad \delta\omega(x) = \sum_{e \in E_x} p(e)\omega(e)$$

(we should recall that d is the coboundary operator acting in 0^{th}-cochains). We easily check that $P - I = \delta d$, and that if p is symmetric with a reversible measure m, then $-\delta$ is the formal adjoint operator of d, and hence P (and $\Delta = P - I$) is symmetric with respect to the inner product defined by

$$\langle f, g\rangle = \sum_{x\in V} f(x)g(x)m(x).$$

In this view, it is natural to say that a function f is harmonic if $Pf = f$, and that $\omega \in C_{-}(E)$ is a harmonic 1-form if $\delta\omega = 0$. Moreover, if the linear operator Δ is invertible, then the kernel function of Δ^{-1} is given by (1).

Graphs themselves are regarded as an analogue of (non-positively curved) manifolds. For example, regular graphs correspond to manifolds with constant negative curvature so that regular trees (simply connected regular graphs) are the corresponding counterparts of unit balls with the Poincaré metric. Harmonic analysis of discrete Laplacians on regular trees associated with simple random walks is well developed as an analogue of the one for hyperbolic spaces (P. Cartier [1]). In this book, regular trees appear, once in a while, as examples for which explicit computations are performed.

Here is a minor remark. In the geometric setting above, the graph X is allowed to have loop edges and multiple edges. In almost all literature, including this book, however, graphs are supposed to have no multiple edges, and hence edges are represented by pairs of vertices. The reason is that, if we are concerned with only sites of a random walker, multiple edges joining two vertices can be reduced to one edge without loss of generality; but if we want to take into consideration which edge is passed by a walker, it is more natural to allow graphs to have multiple edges. An extreme case is a random walk on a bouquet graph, a graph with only one vertex, say o. This being the case, a random walker is always on o, thereby randomness being seen only for (loop) edges the walker passes. By "lifting " the random walker to a regular covering graph of the bouquet graph with covering transformation group G, we obtain a random walk on the Cayley graph associated with G.

Roughly speaking, the contents of this book are made up along the subjects mentioned above. In Chapter I, the author starts with Polya's observation and introduces the basic definitions and concepts in the random walk theory. Various examples and results on recurrence and transience are given. Chapter II discusses the *spectral radius* $\rho = \limsup_{n\to\infty} p_n(x, y)^{1/n} \in (0, 1]$ with which random walks are classified (as the author admits, the name "spectral radius" is misleading since we are not considering the spectral radius of a linear operator acting on a Banach space except for the case of symmetric random walks). It should be noted that ρ^{-1} is the radius of convergence for the power series

$$\sum_{n=0}^{\infty} p_n(x, y)z^n$$

so that, if $\rho < 1$, then the random walk is transient (the converse is not true in general as is seen for the case of $\mathbf{Z}^k$ $(k \geq 3)$). For several classes of infinite graphs, the estimates for ρ are established. The main topic in Chapter III is the local limit formulae. A rough idea is to compare $p_n(x, y)$ with a function of the form $Cn^{-\alpha}e^{-n\beta}\exp(-g(x, y)n^{-1})$. In particular, this chapter includes the local central limit theorem on $\mathbf{Z}^k$ (due to P. Ney and F. Spitzer [6]), Gaussian upper and lower bounds for $p_n(x,y)$, simple random walks on the Sierpinski graphs, and asymptotics of $p_n(x,y)$ for various graphs. A highlight is in the section on the Sierpinski graphs where some fractal properties of the Green function (the kernel function of the operator $(z-P)^{-1}$) are deduced. The most interesting part in this book is in Chapter IV, which treats the ideal boundaries. It contains the exposition of the traditional treatment of Martin and Poisson boundaries in connection with the Dirichlet problem at infinity and the recent developments in the boundary theory of discrete groups and hyperbolic graphs.

The reader might still think that the random walk problems are easy to solve and no complications should appear because the geometric object is just one dimensional and the transition operator is a difference operator. Admittedly, it is true that looking at peculiar phenomena appearing in higher dimensional spaces and in analysis of differential operators is not the business for graphs. Furthermore, the easy construction of examples of graphs possessing desired properties is a big advantage as exhibited in this book. But, there are nonetheless difficulties in the analytic study of infinite graphs which are of the same degree of difficulty as in the case of open Riemannian manifolds. For instance, we do not know, in general, much about the spectra of the transition operators on an infinite regular graph even though its universal covering graph is the regular tree for which the spectra of the transition operators are well-understood, The case of Sierpinski graphs suggests to us that the spectra can be quite complicated. In some cases, exotic phenomena that never appear in "continuous models" may arise for graphs (for example, Cantor spectra for discrete magnetic Schrödinger operators on $\mathbf{Z}^2$, a discrete analogue of the Schrödinger operator with a uniform magnetic field on $\mathbf{R}^2$ whose spectrum consists only of eigenvalues called the Landau level; see M.-D. Choi, G. Elliott and N. Yui [2]).

This carefully written book grew out of the survey paper by the author [9] published in 1994, which became a fundamental reference for the geometric studies of random walks. The organization of the book is well-thought-out. A "Note" which contains information and comments on further studies is found at the end of each chapter. This Note, together with the extensive list of references, is useful to newcomers to the subject who want to know what has already been known. As pointed out by the author in the preface, this book is not self-contained, so that it is intended for graduate students and researchers working in stochastic processes. I think, however, that from the nature of the materials and their presentations, the book is also accessible to undergraduate students and a motivated reader with some basic knowledge of probability and functional analysis. The reviewer has a very high opinion of this book.

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