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Wavelets and operators, by Yves Meyer, Cambridge Studies in Advanced Math., vol. 37, Cambridge Univ. Press, Cambridge, 1992, vii + 223 pp., $49.95, ISBN 0-521-42000-8

A friendly guide to wavelets, by Gerald Kaiser, Birkhäuser, Basel and Boston, MA, 1994, xviii + 300 pp., ISBN 0-8176-3711-7

Although wavelet analysis is a relatively young mathematical subject, it has already drawn a great deal of attention, not only among mathematicians themselves, but from various other disciplines as well. In fact, it is fair to attribute the main driving force of the rapid development of this field to the "users" rather than to the "inventors" of mathematics. To the mathematicians, Fourier analysis has been and still is a very important research area. Its theory is beautiful, its techniques powerful, and its impact on science and technology most profound. However, even as early as the decade of the 1940s, those who used the Fourier approach to analyze natural behaviors were already frustrated with the limitation of the Fourier transform and Fourier series in the investigation of physical phenomena with nonperiodic behavior and local variations. The need for simultaneous time-frequency analysis led to the introduction of Gabor's short-time Fourier transform in 1946 and the so-called Wigner-Ville transform in 1947. But the common ingredient of these two transforms is the sinusoidal kernel in the core of their definitions, so that both high- and low-frequency behaviors are investigated in the same manner and any signal under investigation is matched by the same rigid sinusoidal waveform. In place of the sinusoidal kernel as modulation (for phase shift), a French geophysicist, J. Morlet, introduced in 1982 the operation of dilation, while keeping the translation operation, and developed an algorithm for the recovery of the signals under investigation from this "wavelet transform". It was the mathematical physics group in Marseille, led by A. Grossmann, in cooperation with I. Daubechies, T. Paul, etc., that extended Morlet's discrete version of wavelet transform to the continuous version, by relating it to the theory of coherent states in quantum physics. This was how the notion of the integral (or continuous) wavelet transform was introduced.

The development of the mathematical analysis of the wavelet transform had really not begun, until a year later, in 1985, when the author of the first book under review learnt about the work of Morlet and the Marseille group and immediately recognized the connection of Morlet's algorithm to the notion of resolution of identity in harmonic analysis due to A. Calderón in 1964. He then applied the Littlewood-Paley theory to the study of "wavelet decomposition". In this regard, Yves Meyer may be considered as the founder of this mathematical subject, which we call wavelet analysis. Of course, Meyer's profound contribution to wavelet analysis is much more than being a pioneer of this new mathematical field. For the past ten years, he has been totally committed to its development, not only by building the mathematical foundation, but also by actively promoting the field as an interdisciplinary area of research. The first book under review is the English translation of his first monograph on this subject. In addition to the two subsequent volumes in this three-volume series (the last jointly with R. Coifman), he wrote at least two other shorter monographs on the theory, algorithms, and applications of this subject. D. H. Salinger should be congratulated for an excellent job in translating this classic volume from French to English.

There are many ways to view wavelet analysis. Since the driving force of this field is its applications, particularly to time-frequency analysis, wavelet analysis to the general public is an effective and efficient mathematical tool box for the analysis and synthesis of "signals", where the term "signals" should be interpreted in the widest sense. It must be emphasized, however, that wavelet analysis does not replace Fourier analysis. Instead, it is built on Fourier analysis and enhances it by bringing in the applied and computational aspects of harmonic analysis. Analogous to Fourier analysis where both the Fourier transform and Fourier series are integral parts of this extremely rich field, the subject of wavelet analysis also has the continuous (or integral) and discrete components as well. But unlike Fourier analysis, since both continuous and discrete wavelet transforms are defined on the real-line group, these two components are intimately related. For instance, two functions ψ and $\tilde{\psi}$ in $L^2(-\infty,\infty)$ constitute a pair of "dual wavelets" if the two families $\{\psi_{j,k}\}$ and $\{\tilde{\psi}_{j,k}\}$, where j and k run over the set of all integers, are biorthogonal Riesz bases of $L^2(-\infty,\infty)$. Here, for every function f defined on the real line, the notation $f_{j,k} = 2^{j/2}f(2^j \cdot -k)$ has been used. Hence, the relation between the continuous and discrete wavelet transforms is evident from the observation that for any f in $L^2(-\infty,\infty)$, the coefficients (which constitute the discrete wavelet transform of f) of the series expansion of f in terms of the Riesz basis $\{\psi_{j,k}\}$ are the values of the continuous wavelet transform of f, with the dual wavelet $\tilde{\psi}$ as the "convolution" kernel (or analyzing wavelet), evaluated at the time-scale positions $(k2^{-j}, 2^{-j})$. When some appropriate frequency w_0 of a single f has been identified, the change of scales (say, by 2^{-j} for some integer j) reveals the frequency content at $2^j w_0$ of the signal, with known location near $k2^{-j}$ in the time (or spatial) axis. Furthermore, since the width of $\tilde{\psi}_{j,k}$ narrows or widens as j increases or decreases, the wavelet transform has the so-called zoom-in and zoom-out capability. This is one of the main reasons that wavelet analysis is very useful for time-frequency analysis. The reader is referred to [1] for the point of view from a master of the subject.

Since wavelet analysis is built on Fourier analysis, Meyer's book devotes its first chapter to a brief discussion of the preliminary topics of these two areas, including distributions, the Poisson summation formula, Shannon's Sampling theorem, and the Littlewood-Paley theory. In addition, since Meyer's book is written for mature mathematicians, a section on the related work of Lusin and Calderón is included in the first chapter. On the other hand, since the second book under review is written for readers with only undergraduate mathematics background, its first chapter is devoted to basic mathematics preliminaries, which include even linear algebra and elementary Hilbert spaces and Fourier analysis, while its second chapter is focused on the basic concept of time-frequency localization in signal processing. In this regard, the author, G. Kaiser, tries very hard to demonstrate the "friendliness" of his book, even to those with a very weak background in mathematics and very little exposure to signal processing.

The most powerful tool for the construction of wavelets and for implementation of the wavelet decomposition and reconstruction algorithms is the notion of multiresolution analysis introduced by the author of the first book under review in cooperation with S. Mallat. The second chapter of Meyer's book gives an in-depth discussion of this aspect, even for $L^2(\mathbb{R}^n)$. This includes the orthonormalization procedure and regularity of the scaling functions, and in order to extend to other function spaces, Bernstein's inequalities for $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, are established. The function spaces discussed in this second chapter include the Besov spaces. An example of the Littlewood-Paley multiresolution approximation is also included. As to Kaiser's book the discussion of multiresolution analysis is delayed to Chapter 7, where operational calculus is used to introduce the concept of quadrature mirror filters and, more generally, subband filtering and even wavelet packets. The multiresolution analysis formalism is also emphasized in the second book by considering the complex structure of the operational calculus approach introduced in an earlier work of the author. In this respect, nonacademic engineers who are not used to the formal operational approach may find this chapter difficult to follow.

As mentioned above, the structure of a multiresolution analysis facilitates the construction of wavelets. In the chapter that follows, both in Meyer's book (Chapter 3) and Kaiser's book (Chapter 8), orthonormal scaling functions and wavelets are studied. Here, the spirit of presentation of these two books is again somewhat opposite. Meyer's style is generality, while Kaiser's is friendliness. In Meyer's treatment, after a few examples (including the orthonormal linear spline wavelet and the Meyer wavelets) are constructed, the higher-dimensional setting is treated. Daubechies' wavelets are considered next, and they are used to characterize certain function spaces (mainly the Sobolev and Hölder spaces). In Kaiser's treatment, only the Daubechies wavelets are studied, and this is done by demonstrating a procedure for constructing two examples of the lowest orders. For the more general setting, Kaiser also proposed a new approach based on the statistical concept of cumulants.

In many applications, redundancy (instead of linear independency) is needed. A typical example is human vision. However, stability is always an important issue, and this leads to the concept of frames. Meyer's consideration of frames in Chapter 4 may be considered as a generalization of orthonormal wavelets studied in the previous chapter. However, from the point of view of historical events, it is perhaps more logical to introduce the notion of frames before the study of orthonormal wavelets. After all, Gabor's discretized short-time Fourier transform and Morlet's original formulation of the wavelet transform are frames. This is exactly what Kaiser considers in Chapters 3 to 6, before the introduction of multiresolution analysis and orthonormal wavelets in Chapters 7 to 8. Kaiser's treatment of frames in these four chapters starts with the motivation and precise definition of the integral (or continuous) wavelet transform in Chapter 3 and its reconstruction formula in Chapter 4. The discrete versions of both the short-time Fourier transform and the integral wavelet transform, with emphasis on time-scale/time-frequency analysis, are investigated in Chapters 5 and 6.

Hence, Chapters 1–4 of Meyers' book and Chapters 1–8 of Kaiser's book cover more or less the same basic topics of wavelet analysis, with the distinct difference in that the treatment by Meyer is more abstract and general, while Kaiser's approach is more elementary and puts more emphasis on signal analysis, at times by using very formal operational representations. In the last part of these two books, comprising approximately one third of the entire volume, however, the authors diverge in different directions. Kaiser devotes the remaining portion of his book (Chapters 9–11) to what he calls physical wavelets. This portion consists essentially of the author's own contributions to the field and is intended to introduce wavelets to electromagnetism and acoustics from a theoretical physicist's point of view. In this regard, Part II of Kaiser's book is not written for the typical mathematician or electrical engineer. On the other hand, the last portion of Meyer's book (Chapters 5–6) is written mainly for research mathematicians. In Chapter 5, Meyer discusses the atomic decomposition of the Stein-Weiss $H^2(\mathbb{R}^n)$ space using wavelets and characterizes its dual space $BMO(\mathbb{R}^n)$ in terms of the discrete wavelet transform (or wavelet coefficients). In particular, it is shown that the discrete wavelet transform of a function in $BMO(\mathbb{R}^n)$ satisfies Carleson's conditions, which in turn assure the convergence of the corresponding wavelet series in certain topology. In the final chapter, Meyer applies orthonormal wavelet expansions to study several function spaces, such as those of Hölder, Hardy, Block, and Besov. In addition, the notion of holomorphic wavelets is introduced.

In summary, Meyer's book is the English translation of the first book on an introduction to the mathematical analysis of wavelets. Based on classical Fourier analysis, the author gives a detailed treatment of the construction of wavelets and the application of wavelet series representations to the analysis of the most important function spaces. It is a summary of the development of wavelet analysis in the 1980s and is a classic of this fast-developing field of mathematical analysis. On the other hand, Kaiser's book emphasizes the physics background of wavelet analysis. Its goal is to attract those who wonder what wavelet analysis is about but have no desire to go into much depth in the mathematical development. Since this book provides samples of homework problems, it could be used as a textbook for students in applied mathematics and theoretical physics. However, since its treatment of wavelets in electromagnetics and acoustics is somewhat different from the standard approach to these subjects in electrical engineering, it is perhaps not very warmly welcomed by engineers.

References

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