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Families of automorphic forms, by Roelof W. Bruggeman, Monographs Math., vol. 88, Birkhäuser, Basel and Boston, MA, 1994, x + 317 pp., $99.00, ISBN 3-7643-5046-6

In preparation for this review, I decided to remind myself how I became interested in automorphic forms as an undergraduate. I seem to remember that Eric Temple Bell [1] had something to do with it. Imagine my surprise when I found the following statement on page 333 of [1]: "The subject, elliptic functions, in which Jacobi did his first great work, has already been given what may seem like its share of space; for after all it is today more or less of a detail in the vaster theory of functions of a complex variable which, in its turn, is fading from the ever-changing scene as a thing of living interest."

After wondering how I could have gone into a dead field, I looked up my review [19] of Serge Lang [10]. I think that review made a pretty good case that the subject of automorphic forms (even the special case of modular forms) was still alive in 1980. Now we can also point to the proof by Andrew Wiles of Fermat's Last Theorem via the Shimura-Taniyana conjecture (and as mentioned in a recent "Star Trek Deep Space Nine" episode, the subject may still be alive in the twenty-fourth century).

So then I looked again at Paul Garrett's review [3] of my book [18]. The upshot of the review was a complaint that I had not written a book about group representations and further had skipped the hard details. But I will try not to play a similar trick on the author of the book under review. For once more, the *Bulletin* has chosen a reviewer who would write a completely different book. Unlike [18], the book under review is certainly not full of "extra-mathematical applications and references", nor can it be criticized for leaving out the hard details. This is a treatise for the specialists.

This book does give evidence that E. T. Bell was wrong. We are dealing here with a living field. It is not a subfield of complex analysis. Nor is it a subfield of group representations. Bruggeman's book uses functional analysis and the theory of several complex variables instead. But there is an intersection with most parts of mathematics.

As I have hinted, if you do not know anything about modular forms, this is not the book for you. You should first look perhaps at Svetlana Katok's beautiful introduction to the subject [9], or at [10], or even at [18, Chapter 3]. A brief summary of the entire subject can be fund in the Japan Mathematical Society Encyclopedia article on automorphic forms [8].

To explain a bit about the subject, let us consider a favorite modular form, the cuspidal Maass wave form. It is an $\mathrm{SL}(2,\mathbb{Z})$-invariant eigenfunction for the non-Euclidean Laplacian on the Poincaré upper half plane H, such that $f(z)$ goes to 0 as z approaches the cusp at infinity. More precisely, a cuspidal Maass wave form is a function $f : H \to \mathbb{C}$ which satisfies the following three conditions for all $z \in H$:

1) $$y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f=\lambda f;$$

2) $$f\left(\frac{az+d}{cz+d}\right) = f(z), \quad \text{for all } a,b,c,d \in \mathbb{Z}, \quad ad-bc=1;$$

3) f has vanishing constant term in its Fourier expansion in the $x = \mathrm{Re}(z)$ variable.

At the time of E. T. Bell's book, people such as Erich Hecke studied holomorphic cuspidal modular forms of weight k. This means that 1) was replaced with $f(z)$ holomorphic and the right-hand side of 2) was replaced with $(cz+d)^k f(z)$. Hans Maass changed all this around 1949 (see [13]), and by the mid-1950s Atle Selberg [17] saw how to use Maass wave forms in his trace formula. It turns out that the cuspidal Maass wave forms are much more mysterious than their holomorphic counterparts. For example, the Ramanujan conjecture is still open for the Maass cusp forms, but not for the holomorphic cusp forms. Moreover, there is no nice construction of Maass cusp forms. The tables of eigenvalues and Fourier coefficients in [18] were obtained using computers and are mere approximations to reality.

There are also favorite functions satisfying 1) and 2) but having polynomial growth in y as y goes to infinity and thus not satisfying 3). Eisenstein series span this space. The main object of Part I of the book under review is to give a generalization of work of Y. Colin de Verdière [2]. This leads to the meromorphic continuation of Eisenstein series jointly in weight k and spectral parameter. The subject of meromorphic continuation of Eisenstein series has probably caused a number of forests to be decimated. In the higher rank case (e.g., for $\mathrm{SL}(n,\mathbb{Z})$, $n>2$) you can get some feel for the subject by reading Langlands's review [11] of the book of Osborne and Warner.

Maass's generalization of the concept of modular form in [13] manages to include holomorphic modular forms as well as Maass wave forms. The book under review follows Maass. Then a real analytic modular form f of even weight k satisfies:

1′) $$L_k f = \lambda f, \text{ with } L_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x};$$

2′) $$f\left(\frac{az+b}{cz+d}\right) = \exp(ik\ arg(cz+d))f(z),$$
$$\text{for all } a,b,c,d \in \mathbb{Z}, \quad ad-bc=1;$$

3′) $$f(z) = O(y^a), \text{ as } y \to \infty, \text{ uniformly in } x.$$

If k is not an even integer, as is the case for the Dedekind eta function
$$\eta(z) = e^{\pi iz/12}\prod_{n\geq 1}(1-e^{2\pi inz}),$$
one needs to introduce multipliers into 2′). See page 11 of the book under review. The multiplier system for the eta function can be described explicitly in terms of Dedekind sums (see Lang [10, Chapter 9]):
$$S(d,c) = \sum_{x \bmod c}\left(\left(\frac{x}{c}\right)\right)\left(\left(\frac{xd}{c}\right)\right), \text{ where } ((t)) = \begin{cases} 0, & \text{if } t \in \mathbb{Z}, \\ t - \lfloor t \rfloor - 1/2, & \text{otherwise,} \end{cases}$$
where $\lfloor t \rfloor$ = Floor of $t = [t]$ = greatest integer $\leq t$. Dedekind sums have been of great interest to number theorists and others. The book [21] of Hans Rademacher and Emil Grosswald explains some of the fascination with the subject.

One application of the meromorphic continuation of the Eisenstein family, jointly in weight k and spectral parameter λ, is to obtain distribution results for Dedekind sums. This can be found in Bruggeman, Chapter 13. There is an interesting picture on page 260 showing the distribution of a quantity connected with Dedekind sums.

One must also replace $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ with other discrete subgroups Γ of $G = \mathrm{SL}(2,\mathbb{R})$ such that the volume of $\Gamma\backslash H$ is finite. Examples studied in later chapters are the theta group and the commutator subgroup of Γ. Bruggeman considers subgroups of the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ as well. This makes it easier to deal with multipliers.

The local behavior of cusp forms under variation of the discrete group or the multiplier system has also been considered by Dennis Hejhal [6]. Ralph Phillips and Peter Sarnak [14] consider analytic variation of the Riemannian metric and show that the nonvanishing of certain L-functions at a point implies the annihilation of a cusp form. This has led Sarnak to conjecture that the existence of cusp forms is tied to arithmetic groups such as congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. See Sarnak [15].

Some of these topics are connected with some new life that physicists have been blowing into the subject. We are talking "arithmetic quantum chaos" here (see M. C. Gutzwiller [5] and Sarnak [15]). Quantum physicists and Colin de Verdière asked if the contour maps of cusp forms would localize on geodesics as the eigenvalue λ goes to infinity. Z. Rudnick and P. Sarnak proved this does not happen for congruence subgroups.

D. Hejhal and B. Rackner [7] conjectured that the distribution functions of the cuspidal Maass wave forms for $\mathrm{SL}(2,\mathbb{Z})$ tend to Gaussian with mean 0 and standard deviation $\mathrm{Vol}(X)^{-1/2}$ as λ goes to infinity. You should look at this last reference just for the beautiful color pictures giving the contour maps for Maass cusp forms for $\mathrm{SL}(2,\mathbb{Z})$. Pictures of the level curves are also given. The curves given by $f(z) = 0$ are called "nodal lines", and they show where a non-Euclidean vibrating drum will be at rest during eigenvibrations. The computations were done with a supercomputer.

Many people tody are working on higher rank groups, despite the complicated formulas these groups produce. Yes, the subject is recondite and complex enough that sometimes audiences laugh at you when you produce your beloved formulas. Still, one can derive some joy from seeing progress. I personally have been happy to see the papers of my students Dorothy Wallace [20] and Doug Grenier [4]. They show that in fact one can successfully stick to the classical style of Hecke, Siegel, and Maass in the study of automorphic forms for $\mathrm{GL}(n,\mathbb{Z})$.

Finally, the subject lives in the books of Peter Sarnak [16] and Alexander Lubotzky [12]. These books show that modular forms can be applied to the most amazingly diverse subjects—expander graphs which can be used to produce efficient communications networks and to the question of whether the Lebesgue measure on the n-sphere is the unique rotationally invariant mean on $L^\infty(S^n)$. These problems are reduced to that of estimating the size of Fourier coefficients of modular forms. That is, they are reduced to the Ramanujan conjecture on the size of the Fourier coefficients of holomorphic cuspidal modular forms.

A last question is: Why is this book so expensive? Which leads to another question: Why are my books so expensive? Perhaps there should be a special price for students. I do not know any graduate student who will spend \$50 for a book, much less \$99.

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