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Cobordisms and spectral sequences, by V.V. Vershinin, Translations of Mathematical Monographs, vol. 130, Amer. Math. Soc., Providence, RI, 1993, v + 97 pp., $62.00, ISBN 0-8218-4582-9

Cobordism theory is today one of the basic tools in algebraic topology. The concept of cobordism (or bordism) may be traced back to H. Poincaré and in modern form to L.S. Pontryagin. It is purely geometric. Two closed manifolds V and W of dimension n are (co)-bordant if there exists a manifold U of dimension $(n+1)$ such that

$$\partial U = V \cup W \qquad (\partial \text{ is the boundary operator}).$$

U is then called a bordism between V and W. This notion may be extended to manifolds with various kinds of additional structure such as being oriented, stably almost complex, symplectic, framed, etc. The various classes of manifolds up to the bordism relation form abelian groups which organize to graded rings.

In the literature there is some confusion with respect to the use of the notions cobordism and bordism. Geometrically they mean the same thing. But with respect to the corresponding functors one often uses cobordism theory for the cohomology theory and bordism theory for the homology theory.

The major breakthrough in cobordism theory came with R. Thom [7] when he computed completely the unoriented cobordism ring — showing that it was a graded polynomial ring. His basic idea was to translate the classification problem into a homotopy problem which could be solved. Similarly J. Milnor later on classified manifolds with a stably almost complex structure up to cobordism, showing that the complex cobordism ring is given by

$$\Omega^U = \pi_*(MU) = \mathbf{Z}[x_2, x_4, \dots, x_{2n}, \dots],$$

where MU is the Thom spectrum of the unitary group. Complex cobordism was extensively studied in the 1960's — especially by S.P. Novikov.

In the late 1960's D. Sullivan introduced the idea of studying bordism theory of manifolds with singularities. The idea was reformulated and developed by the reviewer in [3]. It turned out that one could get a natural bordism theory based on manifolds with certain "cone-like" singularities. The idea used in [3] was to remove the singularities and instead study ordinary bordism theory of manifolds with "complicated" boundary structure. If Σ is a set of manifolds and one allows the singularities to be "built up" of cones on elements from Σ, one gets a bordism theory $MU(\Sigma)$ such that

$$\pi_*(MU(\Sigma)) = \pi_*(MU)/(\Sigma)$$

where (Σ) is the ideal generated by the singularity class Σ.

From these theories many other cohomology theories may be derived — for example the extraordinary K-theories called Morava theories, which have led to major advances in homotopy theory. Furthermore this applies also to Brown-Peterson type cohomology theories and elliptic cohomology — which may be constructed by other methods as well. For further information see [5], [6].

Spectral sequences represent important computational tools in algebraic topology which were developed in the 1940's and 1950's. In homotopy theory the Adams spectral sequence turned out to be very powerful, and it was extended to general cohomology and homology theories. For complex cobordism one starts with an initial term

$$Ext_{A^U}(MU^*(X), MU^*(Y))$$

and through spectral sequence computations gets information on

$$\{Y, X\}$$

—the stable homotopy classes. The general idea behind the construction of the Adams spectral sequence is to approximate a space (or a spectrum) by "cohomologically simple" spaces in the form of an Adams resolution which is a geometric analogue to resolutions in homological algebra; see [2], [6].

In the present book the author gives an introduction to (co)-bordism theories with singularities based on the approach in [3]. When a new singularity is added, there is an exact sequence connecting the old and new theory. This leads to a spectral sequence associated with the singularities. Next the author gives a comprehensive discussion of the Adams spectral sequence.

An outstanding problem in cobordism theory is to determine the symplectic cobordism ring

$$\Omega^{MSp} = \pi_*(MSp).$$

The author first studies this problem by the Adams spectral sequence and disproves a conjecture of N. Ray on the vanishing of certain products of elements in the symplectic cobordism ring.

The techniques of cobordism theories with singularities are then used to obtain further results on symplectic cobordism. When the singularity class is taken to be a family of Ray type elements, the corresponding "singularity" theory ring is a polynomial ring over the integers. Furthermore the author shows that the natural spectral sequence associated with the cobordism theories with singularities turns out to coincide with the corresponding Adams spectral sequence. This is a very interesting result and leads to a deeper geometric interpretation of the Adams resolutions.

The book gives a good introduction to cobordism theories with singularities and Adams type spectral sequences. This part of the book is suitable for graduate students in algebraic topology. Some of the sections on symplectic cobordism are more technical and suited for specialists. There is some overlap with [4] on themes — but the texts supplement each other.The book is a welcome addition to the literature on cobordism theory in a time when these ideas find new and interesting applications — as for example in topological quantum field theory; see [1].

Finally, it would have been better if the reference list had been in alphabetical order!

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