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Nonlinear Poisson brackets, geometry and quantization, by N. V. Karasev and V. P. Maslov, Translations of Math. Monographs, vol. 119, Amer. Math. Soc., Providence, RI, 1993, xi + 366 pp., \$170.00, ISBN 0-8218-4596-9

I.

The book deals with two problems:

1. constructing an analog of a Lie group for general Poisson brackets,
2. quantization for such brackets.

Poisson brackets appeared in $\mathbf{R}^n \oplus \mathbf{R}^n$ in the 19th century; Poisson introduced these brackets in analytical mechanics; if f and g are functions of $2n$ variables (p_i, q^i), then

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right).$$

This notion was generalized by Sophus Lie at the end of the 19th century. Then E. Cartan in his famous book *Leçons sur les invariants intégraux* [C] developed Poisson geometry. During a long period, contrary to physicists (Dirac for instance) mathematicians did not study this kind of structure except the symplectic structures (defined by a closed nondegenerate 2-form Ω) such as the symplectic structure on the cotangent bundle to a manifold used in classical mechanics. In 1976, independently A. Kirillov [Ki] and A. Lichnerowicz [Lc] introduced the concept of Poisson manifold. The Poisson bracket on a manifold M turns the space $C^\infty(M, \mathbf{R})$ of functions on a manifold into a Lie algebra which is, with respect to the usual product, a derivation in each of its arguments. So for $f \in C^\infty(M, \mathbf{R})$, the Hamiltonian vector field X_f is defined by the condition $X_f.g = \{f, g\}$ for any $g \in C^\infty(M, \mathbf{R})$. The Poisson bracket may be defined by a bivector field Λ such that $\Lambda(df, dg) = \{f, g\}$. This bivector field Λ satisfies the condition $[\Lambda, \Lambda] = 0$, where $[\ ,\ ]$ is the Schouten bracket. When Λ is nondegenerate, then Λ is the dual of a symplectic form Ω and $\{f, g\} = \Omega(X_f, X_g)$.

The characteristic distribution of a Poisson manifold (*i.e.* the distribution generated at each point by the values at that point of the Hamiltonian vector fields) is Frobenius integrable when Λ is of constant rank (as was proved by A. Lichnerowicz); this distribution is integrable in the sense of Stefan-Sussmann when the rank is not constant (A. Kirillov). The leaves are symplectic manifolds.

The theory of Poisson manifolds has known a great development these last fifteen years, in particular with A. Weinstein [W1, W2] and his school [W.X]. Among the text-books on this subject we mention [L.M] and [V].

The **quantization theory** is a procedure which leads from classical mechanics to quantum mechanics; it associates a quantum system to each classical system, the operation depending upon a parameter $\hbar \in (0, 1]$ with $\hbar = h/2\pi$, h being the Planck constant. The quantum system is reduced to the corresponding classical system when $\hbar \to 0$.

These questions were first studied by physicists (Heisenberg, Dirac, Feynman, H. Weyl).

In classical mechanics, the phase space is a symplectic or Poisson manifold M, the observables are functions on M while in quantum mechanics the observables constitute an algebra C of operators in a Hilbert space $\mathcal{H}$; the bracket $[A, B]$ of the operators is defined by $[A, B] = A \circ B - B \circ A$.

Let H be a Hamiltonian on M and u any observable; the evolution of $u_t = u(x_t)$ (with $\dot{x}_t = X_H(x_t)$) is given by $\dot{u}_t = \{H, u_t\}$. If $\widehat{H} \in C$ is an operator on $\mathcal{H}$, for any $\widehat{u} \in C$, the evolution of $\widehat{u}$ is given by $\dot{\widehat{u}}_t = [\widehat{H}, \widehat{u}]$.

So quantization leads from Poisson manifolds to noncommutative algebras [Co].

A long time ago, Bohr and Sommerfeld obtained quantization rules, giving an explanation of the spectrum of the hydrogen atom.

Then many mathematicians tried to obtain an intrinsic formulation of these results, independent of the choice of coordinates.

Independently Kostant [Ko] and Souriau [S] introduced the **geometric quantization** on a symplectic manifold M; the procedure goes through prequantization i.e. the use of a complex line bundle $K \to M$; the existence of K requires that the cohomology class of the symplectic form be an integral class. Similar results can be proved for general Poisson manifolds with the consideration of the class of Λ in the Poisson cohomology introduced by A. Lichnerowicz.

Deformation quantization has been initiated by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [B.F.F.L.S]. It is a formal deformation of the Poisson bracket by use of a $*$-product

$$\{f, g\}_t = \frac{1}{2t}(f * g - g * f) = \{f, g\} + \sum_1^\infty t^{2p} \mu^{2p+1}(f, g),$$

where μ is a bidifferential operator.

The existence and classification of these $*$-products have been studied also by Arnal, Cahen, Cortet, Gutt, Lecomte and de Wilde [DW.L].

Asymptotic quantization has been initiated by Karasev and Maslov (see the second part of this review).

The problem of quantization suggested independently to A. Weinstein [W2, W3] and M. Karasev [Ka] the introduction of **symplectic groupoids**. The theory of groupoids in differential geometry was introduced in 1950 by C. Ehresmann [E] in his paper on connections (where he defined the groupoid of paths, the gauge groupoid associated to a principal bundle). A groupoid is a set Γ equipped with a partially defined associative multiplication such that any $x \in \Gamma$ possesses a unique right unit $\alpha(x)$ and a unique left unit $\beta(x)$, as well as a unique inverse x^{-1} such that $x^{-1}x = \alpha(x)$, $xx^{-1} = \beta(x)$; the product xy is defined if and only if $\alpha(x) = \beta(y)$. A differentiable manifold Γ endowed with a groupoid structure is a Lie groupoid if the base Γ_0 (set of units) is a submanifold of Γ, the mappings α and β are submersions, the multiplication $(x, y) \mapsto xy$ is differentiable and the inversion $x \mapsto x^{-1}$ is a diffeomorphism. For instance if we consider on a manifold M a pseudogroup Θ of diffeomorphisms (i.e. a collection of local diffeomorphisms such that the composition of two diffeomorphisms of Θ, when it exists, is also an element of Θ), then the set of germs (or of k-jets) of such diffeomorphisms is a Lie groupoid.

A symplectic groupoid is a Lie groupoid equipped with a symplectic structure such that the multiplication is a Poisson mapping and the manifold Γ_0 of units is a Lagrangian submanifold. One of the main properties of symplectic groupoids is the following: the base Γ_0 is endowed with a Poisson structure such that α (resp. β) is a Poisson (resp. anti-Poisson) morphism. Conversely the problem of integration of a Poisson manifold P is the following: find a symplectic groupoid Γ such that P is the base. The problem leads to topological obstructions (see A. Weinstein and his students [W.X], A. Weinstein and his coworkers [C.D.W], P. Dazord and G. Hector [D.H], C. Albert and P. Dazord [A.D] as well as the Russian school). The integration is always possible if we restrict ourselves to a local groupoid (roughly it is a manifold with a local multiplication, for which the condition $\alpha(x) = \beta(y)$ is necessary but not sufficient for xy to be defined).

II.

The book under review utilizes some of the results which have been published by the authors in previous papers. This book is very interesting; it contains a wealth of original ideas, but it needs an effort from the reader. As happens rather often for publications full of new ideas, it is not always written very clearly; nevertheless it contains detailed proofs.

The book contains four chapters and two appendices. Chapters I and II deal with differential geometry, Chapters III and IV deal with semi-classical approximation. The appendices deal with 1) noncommutative analysis, 2) calculus of symbols and commutation relations.

Chapter I is devoted to Poisson manifolds. The authors study Poisson brackets related to Lie groups, in particular they study Poisson Lie groups and their bialgebras. Then the problem of reduction is discussed with the introduction of the notion of bifibration (this notion in the case of a symplectic manifold is equivalent to that of "symplectically complete" foliation introduced by the reviewer [Lb] and of "dual pair" introduced by A. Weinstein [W1]). This leads to the Dirac bracket and its generalization. At last the problem of perturbations and cohomology of Poisson brackets are studied.

Chapter II deals first with symplectic groupoids and the integration of a Poisson manifold by a local symplectic groupoid. The consideration of "Cartan structures"on a manifold endowed with 1-forms satisfying generalized Maurer-Cartan equations leads to the notion of so-called "finite-dimensional pseudogroup". This notion is different from what is usually called a pseudogroup of transformations (see above), but in Chapter IV section 2.2 the authors use the word "pseudogroup" in the usual meaning.

Roughly speaking a "finite-dimensional pseudogroup" G over a Poisson manifold $\mathcal{N}$ is a manifold G with a distinguished point e together with a map $\Phi : G \times \mathcal{N} \to \mathcal{N}$ such that $\Phi(e, s) = s$ for any $s \in \mathcal{N}$, the composition law and the inversion depending upon the manifold $\mathcal{N}$; moreover the composition law must satisfy conditions such that 1) the orbits of the action coincide with the symplectic leaves of the Poisson structure, and 2) the manifold $G \times \mathcal{N}$ is endowed with a groupoid structure. The notions of right and left vector fields on G may be defined as well as the notion of fundamental vector field on $\mathcal{N}$, inducing structure functions. When $\mathcal{N}$ is simply connected and possesses a flat linear connection, then $G \times \mathcal{N}$ is endowed with a symplectic groupoid structure.

These notions are useful for the problem of quantization, but the development of these questions is interesting in itself, and the book may provide new ideas to differential geometers.

Chapter III is concerned with semi-classical approximation in the phase space $\mathbf{R}^{2n}$; it is a technical chapter where the authors study the wave packets (elements of a subspace of the space of mappings from the set of parameters $\hbar \in (0,1]$ into the Schwartz space $S(\mathbf{R}^n)$ of smooth rapidly decreasing functions), the quantum density and the front of such wave packets. Then, using the Maslov index, they define and construct the intertwining operator between classical and quantum variables. They study the quantization of solutions to Hamiltonian systems.

Chapter IV deals with the asymptotic quantization of symplectic manifolds and more generally of Poisson manifolds with degenerate Λ.

This theory, due to the authors, stands between geometric and deformation quantization. With each function f is associated a pseudo-differential operator $\widehat{f}$ (whose symbol is f) acting on a "sheaf of wave packets" Γ. This object Γ is a kind of fiber bundle, but the gluing mappings satisfy the cocycle conditions mod $O(\hbar^k)$. The operator $\widehat{f}$ satisfies the conditions

$$\widehat{f}\widehat{g} = \widehat{fg} + O(\hbar)\,,$$
$$[\widehat{f},\widehat{g}] = i\hbar\widehat{\{f,g\}} + O(\hbar^3)\,.$$

The paper [K.M] contains some mistakes; P. Dazord and G. Patissier [D.P] have given a correct version of the construction which is utilized in the book under review. For instance the condition on the existence of the asymptotic quantization of a symplectic manifold (M,Ω) is

$$\frac{[\Omega]}{2\pi\hbar} - \frac{1}{2}C_1(M,\Omega) \in H^2(M,\mathbf{Z})\,,$$

where $C_1(M,\Omega)$ is the first Chern class and $[\Omega]$ the class of Ω in the de Rham cohomology. If $C_1(M,\Omega)=0$, the above condition is the prequantization condition.

The quantization of a general Poisson manifold (M,Λ) leads to the quantization of a symplectic manifold when there exists a "finite dimensional pseudogroup" G over M (in the sense of Chapter II) such that $M\times G$ is endowed with a symplectic groupoid structure.

The authors explain their theory of quantization in studying examples (sphere, sphere with horns, torus). They deal with quantization of two-dimensional surfaces.

To conclude, this book is a very stimulating contribution to a field of research in full extension. May the reviewer point out some deficiencies in the presentation:

1. there are no subject or notation indexes,
2. the list of references is rather "untidy".

For instance two books by E. Cartan are listed under numbers 75 and 165; moreover these books are not concerned with the subject; the correct reference is *Leçons sur les invariants intégraux* ([C] in our bibliography).

Our references contain only one previous paper by Karasev [Ka] and another by Karasev-Maslov [K.M]. The others are to be found in their book.

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