BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, January 1996

Algebraic L-theory and topological manifolds, by A. A. Ranicki, Cambridge Tracts in Math., vol. 102, Cambridge Univ. Press, Cambridge, 1992, 358 pp., $69.95, ISBN 0-521-42024-5

Topological manifolds are very simple objects to define: they are separable metric spaces that are locally homeomorphic to Euclidean space. In other words, these are objects which are locally coordinatized but not globally so.

Smooth manifolds require additional effort in definition; one must choose coordinate charts and insist on compatibilities among them. This makes it possible to define the class of smooth maps between manifolds. Sard's theorem, the implicit function theorem, and other tools of calculus are now available, and one can begin to explore the geometry of manifolds. For instance, if one starts with an arbitrary map $M \to \mathbb{R}$ and takes a suitable generic approximation, one obtains a smooth function with isolated critical points and nonvanishing Hessian at each of these points. Applying the Morse lemma (see [Mi2]) to this function gives what is called a "handlebody decomposition" for the manifold. This is an essentially combinatorial object that can be successfully manipulated. Moreover, the calculus of such manipulations can be algebraicized in a rich and beautiful fashion.

Ultimately, transversality and handlebody decompositions together with the techniques of surgery and healthy doses of homotopy theory and algebra/number theory give us our modern theory of smooth high-dimensional manifolds as developed by Smale, Kervaire-Milnor, Browder, Novikov, Sullivan, Wall, and many others. A rather good explanation of this development can be found in Wall's book [Wa1] (or Browder's [B] for the simply connected case), augmented by [O, H] for the necessary algebra and [MM] for the homotopy theory. (A useful introduction may be found in [KO].) Rothenberg's review [RO] gives an historical overview of this work.

For topological manifolds the analogous geometric tools do not seem to be available. Moreover, there are a number of very useful generalizations of manifolds, such as orbifolds, where one can prove that these tools do not exist: transversality is obstructed[1], and the analogs of handlebody structures generally do not exist. Even when they do exist, they are not unique in the usual sense. (See e.g. [MR], [Q1], and [CS1].)

Despite these difficulties, there is a beautiful and remarkably comprehensible classification theory of topological manifolds and their generalizations which is in many ways simpler than the smooth theory. It turns out that the tool theorems remain true for topological manifolds but that the implications of these theorems are somewhat different. (For the other generalizations, as I mentioned, even the tools are gone.) One sees this immediately in the classification of manifolds homotopy equivalent to the sphere: smoothly there are very many, and their number is a quite irregular function of dimension. Topologically the sphere is the only such manifold (except perhaps in dimension three). Formally this is reflected in the classifying space of topological bundles being rather different from Grassmanians. The extra

The reviewer was partially supported by a National Science Foundation grant.
[1]Not that the obstructions have ever been adequately analyzed.

complexity in the classifying space for topological bundles precisely counters the complexity of the smooth homotopy spheres.

Here would be an appropriate place for me to describe the long and tortuous path to this classification, how the missing tools are provided for the topological category and the surprising role that understanding manifolds homotopy equivalent to the n-torus (provided by [HS] and [Wa2]) plays, but developing this theory is the focus of the book [KS]. Instead I will focus on what the answer is.

For that I need notation. Let M be a manifold, perhaps with boundary. $S(M)$ will denote the set of pairs (N, f), where f is a (simple) homotopy equivalence from a manifold N to M. We view (N, f) as being trivial if f is homotopic to a homeomorphism. If there is boundary, we assume that all maps restrict to homeomorphisms on the boundary.

Theorem. *If $n = \dim M$ is at least $5, S(M)$ is an abelian group. If $\partial M \neq \emptyset$, it can be computed via the following long exact sequence*:

$$\cdots \to L_{n+1}(\pi_1 M) \to S(M) \to H_n(M; L) \to L_n(\pi_1 M).$$

Here the groups $L_n(\pi_1 M)$ are functors of the fundamental group (and orientation character, which we have suppressed from the notation) and $H_n(M; L)$ is the nth term in a generalized homology theory.

The hypothesis is that $\partial M \neq \emptyset$ can always be achieved by removing a small open ball from M and is only a very small nuisance. One would otherwise have to replace the homology term by a slightly more complicated one than I have in mind. (It would also be homological, but it would differ slightly depending on n.)

The groups $L_n(\pi_1 M)$, called surgery groups, are generalizations of Witt groups of quadratic forms over the integral group ring of $\pi_1 M$. The group $H_n(M; L)$ is not too mysterious: at 2 it is a sum of (ordinary) homology groups, and away from 2 it is KO-theory (i.e. the theory dual to real vector bundles). The map $H_n(M; L) \to L_n(\pi_1 M)$ is remarkable: it relates characteristic class theory (i.e. tangential information) to underlying algebraic topological invariants of the space. When M is simply connected and $n = 4k$, this map encodes the Hirzebruch signature theorem (see [Hi]). It is traditional, although mystifying to beginning surgeons and, even more so, to specialists in other fields, to call this map the assembly map.

I should remark that just as the modern approach to the Hirzebruch signature theorem sees it as a special case of the Atiyah-Singer index theorem, there is a close analogy between many aspects of surgery theory (and, in particular, the assembly map) and more modern index theory; see [Ros, W1, Ob] for more details.

The connection between geometry and algebra implied by the assembly map can be achieved in one of two ways: given a geometric definition of L-groups (already done in Wall's book) or giving more algebraic treatments of the geometry.[2] Ranicki has been an active advocate for this second approach, and it has much to recommend it. Extending earlier work of Mishchenko, he has shown that the Witt groups can be thought of as "cobordism classes of algebraic Poincaré complexes". These flexible objects can sometimes be chopped up as manifolds can, and they can be glued together. On the other hand, since they are defined by pure algebra, one can use all of the standard algebraic tools, like localization and completion, to study them.

Note that here the characteristic class theory is viewed as being a theory of homology characteristic classes. Of course, for manifolds one generally has Poincaré

[2]In index theory this is provided by the Ext interpretation of K-homology; see e.g. [D].

duality, and the choice of homology over cohomology is therefore a matter of taste; but (1) the orientability of manifolds for the relevant homology theory is quite a deep matter (due to Quinn and Ranicki), and (2) the translation of the data into homology makes functoriality possible, as the L-groups are covariantly functorial, while cohomology theories are contravariant. Thus, a rather deep aspect of this sequence is that it enables one to make $S(M)$ a covariant functor of M (with respect to orientation-preserving maps between manifolds whose dimensions only differ by multiples of 4). Yet more remarkable is the fact that this sequence has a 4-fold periodicity (due to Siebenmann, with a correction by Necas)—analogous to Bott periodicity (see [B1])—at least if M has nonempty boundary $S(M) \cong S(M \times D^4)$. This is one place where the topological theory is simpler than the corresponding smooth theory—functoriality is not known in the smooth category, and periodicity is hopelessly false.

To summarize, I have presented a formal exact sequence that computes the manifolds homotopy equivalent to a given one up to homeomorphism. I have asked you to believe this summary, despite the fact that the tools that could prove it in the smooth case are absent and the fact that it is not even true, as stated, in the smooth category. What happens in the smooth case is that there is in fact a surgery sequence, except that the term replacing $H_n(M; L)$ in the sequence is of the form $[M\colon\ F/O]$, where F/O is a very mysterious space, so that calculations are hard. Furthermore, the functor $M \to [M\colon\ F/O]$ is contravariant—a defect which does not allow for producing a functorial surgery exact sequence, since the L-groups are covariant functors of the groups.

This understanding of the subject would be enough for someone who is just interested in the problem of computing $S(M)$ in the topological category. (And occasionally, one can get smooth results by comparison; see [W2].) All of the terms in the surgery exact sequence have been carefully studied, and one can appeal to the literature I mentioned before.[3] But that would really miss another key aspect of the whole subject. Notice the notation: The homology theory is with coefficients labeled by the same symbol as the surgery group L. This is not abuse of notation. From the modern point of view, the homology term is essentially the same sort of object as the algebraic term. In other words, the homology term is a group built up out of something like a self-dual complex of modules "spread over M", while the algebraic term is described as the same thing, except that it is concentrated at a point.

In other words, *surgery theory asserts that $S(M)$ is a measure of the difference between "local" and "global" L-groups.* This local-global aspect of the answer has been of profound significance in a number of more recent theoretical investigations. (In other words, it is good for proving theorems even if it doesn't usually help that much for doing calculations.)

I should remark that much work done since the mid-1980s has clarified the sense in which the homology term is to be viewed as "local L-theory". See e.g. [Q2, Y, FP]. (This is a major part of the subject of "controlled topology".) But deep and important as this work is, and while necessary for many other important applications, its understanding is not critical for a suitably Olympian understanding of the statement of classification as a measure of local-global mismatch.

[3] It seems worth noting that the calculations tend to be quite algebraic for finite groups, very geometric for torsion-free groups, and an intricate mix for infinite groups with torsion.

This view of classification, bereft of complicated classifying spaces, has now spread to many other problems in high-dimensional geometric topology. The simplest one to state, although the most recent, is the classification of homology manifolds (see [BFMW, W3]. These are spaces defined by a local homological condition (namely, that $H_*(X, X-p) \cong H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$, which is the local cause of Poincaré duality). These can be classified up to s-cobordism exactly by the sequence asserted by the classification theorem above! In other words, by enlarging the class of spaces considered, one can avoid the $\partial M \neq \emptyset$ condition: it turns out that one had all along been requiring an unnatural additional local condition (namely, manifoldness) not required by the problem and that this extra condition only becomes natural for spaces with boundary, when the boundary condition forces it.

Here the homology class associated to X is the result of the methods of controlled topology applied to the self-dual sheaf given by the singular chain complex of X (the self-duality following from the homology manifold condition). This gives a roundabout but topologically invariant definition of rational Pontrjagin classes, the topological invariance of which, given the usual smooth definition, was the occasion of Novikov's Fields Medal.

On the other hand, there are sources of self-dual sheaves other than just homology manifolds. Intersection homology provides another important source (see [GM] II). Self-dual sheaves can be pushed forward and occasionally pulled back, and formulae for these have been quite useful in problems related to characteristic classes of singular varieties and lattice point counting problems and Euler-MacLauren formulae (see [CS2, CS3, Sh]).

Another place where assembly maps play a large role is in the theory of rigidity. The fact about tori mentioned before is that PL homotopy tori have finite sheeted covers that are PL homeomorphic to the torus. Topologically, there is no need for passing to covers, and indeed in all known examples where one can understand the manifolds homotopy equivalent to an aspherical one, there is only one! This statement is called the Borel conjecture, and it has been verified in many cases (see [FJ] for a survey). Critical for its understanding is the algebraic reformulation asserting that for aspherical manifolds, an assembly map is an isomorphism. Much of the work on this problem is devoted to proving that the assembly map is an isomorphism for all *torsion free* groups. This extension would be quite hard to phrase purely geometrically but is clearly central to the problem. In addition, this algebraic formulation also suggests analogs in K-theory and for other functors—generalizations that are intrinsically interesting and sometimes shed light on the original question.

Yet another example of the power of the assembly point of view on classification is in the theory of orbifolds (and stratified spaces; see [W1]). For simplicity let us consider quotients of manifolds by finite group actions (rather than objects that are just locally such). For actions of odd-order groups that are locally smooth, Madsen and Rothenberg [MR], have, by a tour de force of geometric reasoning combined with algebraic calculations, shown that all of the tool theorems of the smooth (unequivariant) category have counterparts. This gave a surgery exact sequence of cohomological type (as in the classical smooth category) and implied, for instance, that topologically conjugate representations of odd-order groups are linearly conjugate. (This had also been proven independently in [HP] by a less fundamental but more direct approach.)

On the other hand, in 1982 already Cappell and Shaneson [CS1] had shown that topological conjugacies are quite common for representations of even-order groups, so one knew that the tool theorems could not be pushed to this generality.

We now understand the reason. When algebraic geometers began defining characteristic classes for singular varieties (see [Mp, GM], they discovered that what generalizes well are not the characteristic cohomology classes, but rather their Poincaré duals. For nonmanifolds the difference is quite serious, because often the homology classes cannot be pulled back to cohomology.

It turns out that the same is true in topology. We now understand that the tools used in the smooth category such as transversality and handlebody structures were a gift and that for a host of other problems where one wants classification they do not exist. Nonetheless, the theory obtained by formally replacing the cohomology classes by homology exists, has good geometric meaning, and is more widely available. Cohomological theories are available only when transversality is. Thus, for even-order orbifolds, it still turns out that there is a homological form of equivariant surgery that remains valid; of necessity, it must be proven by more complicated methods circumventing the lack of familiar tools. (Both the theorems of [CS1] and of [HP, MR] can be obtained from this homological theory.)

Now let me turn to the book under review. Its main goal is to describe the theory of topological manifolds as set out above. (It does not go as far as explaining how the homological viewpoint avoids the sometimes insurmountable difficulties involved with transversality; the author mainly deals with a situation where transversality is true.) This involves defining the L-groups, the homology theory, and assembly map and relating all of this to geometry. The algebraic approach to this, due to the author, who is professor of algebraic surgery at Edinburgh, is presented here in complete detail; this is a very valuable addition to the literature. As I mentioned before, assembly maps arise in several areas of mathematics, and the detailed explicit treatment given here will no doubt be widely studied.

The surgery exact sequence, as presented here, does depend on already knowing the smooth case, and at critical points, one invokes (in effect) the topological s-cobordism theorem and topological transversality in deducing the beautiful local-global form from the smooth case. Thus, for the geometry behind the theory, the reader will have to turn elsewhere. (Some sources for this kind of material are, besides the classic works of Milnor, Browder, and Wall, the recent volumes of [KO, W2] and unpublished notes, which I believe are available from the authors, by Milgram-Ranicki on surgery theory and Ferry on the geometrical foundations of the topological category.) However, the algebra, which is often omitted in more geometric treatments and which is critical to many of the interactions between surgery and others parts of mathematics, is here treated in great detail.

Besides this central goal, Ranicki describes a few of the directions in which the theory is applied. He gives a rapid sketch of some of the ideas involved in computing for finite groups, the theory of splitting homotopy equivalences, algebraic reformulations of the celebrated Novikov and Borel conjectures, and the statements of the results for homology manifolds mentioned above. I found these chapters interesting and clear, and I think that a reader learning the subject from this book would be motivated to turn also to the research literature on these topics, beginning with the papers that Ranicki cites.

I think that the geometric topology student who wants to learn surgery theory would still do best by starting with the classic papers and books. After one has a sufficient geometrical maturity, the treatment here could be better swallowed. The organization, with Part I being "Algebra" and the topology part only beginning after some 170 pages, would stymie such a student; but nonetheless, this material is important, and the student must somehow ultimately come to terms with it. On the other hand, someone who is more algebraically oriented or who is already familiar with assembly maps from another point of view, either from, say, algebraic K-theory, or operator theory, or even geometrically (seeing surgery problems by gluing simplices of them together!) will find the current treatment insightful and illuminating. I welcome its arrival to the literature (as I do Ranicki's other efforts aimed at covering related material in [R1, MiR]).

Final Remark. The author of this work has made available by anonymous ftp a list of errata that he plans to update as necessary. These can be obtained on the WWW: `http://www.maths.ed.ac.uk/people/aar/`. This seems to me a most welcome application of computer technology.

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Shmuel Weinberger
University of Pennsylvania
E-mail address: shmuel@math.upenn.edu