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CRAIG HUNEKE
PURDUE UNIVERSITY

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Geometry of Banach spaces, duality mappings and nonlinear problems by Ioana Cioranescu, Kluwer Academic Publishers, Dordrecht, 1990, 260 pp., \$99.00. ISBN 0-7923-0910-3.

What can replace the inner product in non-Hilbert Banach spaces? To answer this natural question we may proceed as follows. Let X^* be the dual space of a real Banach space X, and denote the norms of both X and X^* by $|\cdot|$. For each x in X define

$$J(x) = \{x^* \in X^* \colon (x, x^*) = |x|^2 = |x^*|^2\}.$$

This weak-star compact convex subset of X^* is always nonempty by the Hahn-Banach theorem, and the mapping $J\colon X \to 2^{X^*}$ is called the normalized duality mapping of X. For x and y in X we now define two semi-inner-products by

$$(y, x)_+ = \max\{(y, x^*) \colon x^* \in J(x)\}$$

and

$$(y, x)_- = \min\{(y, x^*) \colon x^* \in J(x)\}.$$

Equivalently,

$$(y, x)_+ = \lim_{t \to 0+} (|x + ty|^2 - |x|^2)/(2t)$$

and

$$(y, x)_- = \lim_{t\to 0-} (|x+ty|^2 - |x|^2)/(2t).$$

These semi-inner-products have some, but, in general, not all the properties of an inner product. The duality mapping is single-valued, and $(y, x)_+ = (y, x)_-$ if and only if X is smooth. When X is a Hilbert space that is identified with its dual, the duality mapping becomes the identity, and $(y, x)_+ = (y, x)_-$ coincides with the inner product of X. Consider, for example, the sequence spaces l^p, $1 \le p < \infty$. For $1 < p < \infty$ the duality mapping of l^p is single-valued, $J(0) = 0$, and

$$(Jx)_j = (|x_j|^{p-1} \operatorname{sgn} x_j)/|x|^{p-2}$$

for all $x \neq 0$ and $j = 1, 2, \ldots$. In this case

$$(y, x)_+ = (y, x)_- = \left(\sum_{j=1}^{\infty} y_j x_j |x_j|^{p-2}\right) / |x|^{p-2}.$$

If, however, $p = 1$,

$$(Jx)_j = \begin{cases} (\operatorname{sgn} x_j)|x| & \text{if } x_j \neq 0, \\ [-|x|, |x|] & \text{if } x_j = 0, \end{cases}$$

and J is set-valued. In this case $(y, x)_+ = (y, x)_-$ if and only if $x = 0$ or $y_j = 0$ whenever $x_j = 0$.

Sometimes it is advantageous to use more general duality mappings. Let $\varphi\colon [0, \infty) \to [0, \infty)$ be continuous and strictly increasing with $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The duality mapping of X with gauge function φ is defined by

$$J_\varphi(x) = \{x^* \in X^*\colon (x, x^*) = |x|\varphi(|x|) \text{ and } |x^*| = \varphi(|x|)\}.$$

This duality mapping is the subdifferential of the convex function $\Phi(|x|)$ where

$$\Phi(t) = \int_0^t \varphi(s)\, ds.$$

While the normalized duality mapping of the sequence spaces l^p, $1 < p < \infty$, is demicontinuous, the duality mapping J_φ with $\varphi(t) = t^{p-1}$ is, in fact, weakly sequentially continuous.

Since duality mappings were introduced by Beurling and Livingston [4] (see also [12, 13, 14, 15]) and the early work of Browder [5], Asplund [2], Kato [11], and others, they have continued to be a very useful tool in both linear and nonlinear functional analysis (see, for example, the papers [19, 16, 23, 17, 1] and the books [3, 6, 10]). One of the main reasons for this is their close connection with accretive and monotone operators. Recall that a set $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ is said to be accretive if

$$|x_1 - x_2| \le |x_1 - x_2 + r(y_1 - y_2)|$$

for all $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$, and all positive r. Equivalently, the operator A is accretive if

$$(y_1 - y_2, x_1 - x_2)_+ \ge 0$$

for all $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$. If, in addition, $R(I + A) = X$, then A is called m-accretive. Such operators are important because they govern many nonlinear evolution equations and, in particular, generate nonlinear nonexpansive semigroups. A subset $M \subset X \times X^*$ is called monotone if

$$(y_1^* - y_2^*, x_1 - x_2) \geq 0$$

for all $x_i \in D(M)$, $y_1^* \in Mx_i$, $i = 1, 2$. It is said to be maximal monotone if, in addition, there is no proper monotone extension of M. Duality mappings are always maximal monotone. If X is reflexive and both X and X^* are strictly convex, then M is maximal monotone if and only if $R(J + M) = X$. Such operators are important in the study of nonlinear elliptic boundary value problems and in optimization theory.

In Hilbert space the class of accretive operators coincides with the class of monotone operators. Outside Hilbert space, the properties of accretive operators are often determined by the continuity properties of duality mappings that are, in turn, often equivalent to differentiability properties of the norm. Since a monotone operator remains monotone even if the original norm of X is replaced by an equivalent one, renorming theorems are helpful in monotone operator theory. We also note [18], for example, that the normalized duality mapping of a Banach space is strongly monotone if and only if the space is uniformly convex with a modulus of convexity of power type 2. Thus it is not surprising that the geometry of Banach spaces plays such an important role in the study of nonlinear operators.

In the book under review, a completely rewritten and expanded version of [7], the author uses duality mappings to link several topics in linear and nonlinear functional analysis. She begins with some convex analysis and continues with a study of the properties of duality mappings in various Banach spaces. Several renorming results are then followed by a chapter devoted to degree theory for A-proper mappings. Finally, there is a discussion of monotone and accretive operators, as well as nonlinear semigroups. Each of the six chapters also contains exercises and bibliographical comments. Since the book is quite self-contained, it can serve as a supplementary text for a basic course in nonlinear analysis. Unfortunately, it is marred by numerous misprints and inaccuracies. Here is a small sample from the first few pages: on p. 6, line 18, " $-F(x)$ " is missing from the definition of the directional derivative; on p. 14, the proof of Corollary 2.7 should refer to Proposition 2.5, Theorem 2.6, and Corollary 1.20; on p. 17, line 3, the definition of the conjugate function is incorrect; an inequality sign is missing on p. 20, line 16, and an equality sign is missing on p. 21, line 27; the last line on p. 21 should refer to Corollary 2.7. Also, the book is not as up-to-date as one might hope. On p. 207, for example, the author states that except in Hilbert space, it is not known whether every nonexpansive semigroup (on a closed convex subset of X) is generated by an accretive operator via the exponential formula. As a matter of fact, this is known to be true whenever the space X is reflexive with a uniformly Gâteaux differentiable norm [20]. Moreover, if, in addition, the norm of X^* is Fréchet differentiable, then there is a bijective correspondence between m-accretive operators in $X \times X$ and semigroups on nonexpansive retracts of X.

We conclude by mentioning that several recent applications of duality mappings can be found in [21, 8, 24, 25, 22, 9].

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SIMEON REICH
THE UNIVERSITY OF SOUTHERN CALIFORNIA
THE TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY