some day write a longer book that will have much more background material and display his excellent expository abilities again.

CATHLEEN S. MORAWETZ
COURANT INSTITUTE OF MATHEMATICAL SCIENCES,
NEW YORK UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 2, April 1992

0273-0979/92 $1.00 + $.25 per page

Homological Questions in Local Algebra, by Jan R. Strooker. Cambridge University Press, Cambridge 1990, 307 pp., $34.50. ISBN 0-521-31526-3

In the late 1950s the use of homological methods revolutionized commutative noetherian ring theory. Maurice Auslander, David Buchsbaum, Jean-Pierre Serre, and others used homological methods to solve several open problems in commutative algebra. New questions were suggested by their work and, in some cases, were conjectured by them. These questions became known as the homological conjectures; other problems that grew from the original list were later added.

Perhaps the most famous problem solved during this time was the proof, due to Auslander and Buchsbaum [AB], that regular local rings have the unique factorization property. Regular local rings are the generalization of the local rings at smooth points in algebraic varieties. They are defined by the condition that the minimal number of generators of their unique maximal ideal is equal to the dimension of the ring (see Definition 1.2). It is always true that the minimal numbers of generators of the maximal ideal is at least the dimension of the ring (see e.g., Krull's theorem below).

During the 1960s some progress was made on the homological conjectures. Nonetheless, there was not a great deal of progress until the late 1960s. (Actually, many were not stated until the 1970s). With hindsight, this was because the techniques to solve them were not in place. The proofs of most of these conjectures (for arbitrary noetherian rings that contain a field) require a method called "reduction to characteristic p," and exploit the Frobenius endomorphism of a ring of characteristic p that sends an element to its pth power. To achieve this reduction an important theorem of Michael Artin is needed, called the Artin Approximation Theorem [Ar], which was not proved until the late 1960s (although special cases were studied earlier by Lang and also by Greenberg). This method of reduction essentially allows one to give proofs for rings of characteristic p, then to claim the validity of the theorems for rings that contain the rationals provided the statements of the theorems can be expressed "equationally." On the other hand, the proofs of many of the homological conjectures could have been done much earlier in characteristic p and for finitely generated algebras over fields.

All rings in this review will be commutative, noetherian, and with identity. All modules will be unital. A finitely generated free complex over R is a finite complex of finitely generated free R-modules,

$$(1.1) \qquad \mathbf{F}: 0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to 0.$$

Such a complex is *acyclic* if the only nonzero homology occurs at the 0th spot. In this case the complex is said to be a (finite free) resolution of $M = H_0(\mathbf{F})$.

A basic theorem, arising from work of Hilbert and Serre, gives that every finitely generated module M over a polynomial ring $R = k[x_1, \dots, x_n]$ (with k a field) has a finite resolution by finitely generated free modules. Intuitively, such a resolution can be thought of as "unwinding" the module structure of M. A free R-module is the simplest of R-modules; as the length of the resolution grows so, in general, does the complexity of the module. After choosing bases for the free modules in a resolution of M, one obtains matrices giving the maps in the resolution. The ideals generated by the minors of these matrices play a key role in understanding the structure of the module M. The simplest nontrivial case of this phenomenon is well known to beginning graduate students in algebra; namely, a finitely generated module over a PID is determined up to isomorphism by its invariant factors, which are nothing more than the ideals of minors in a presenting matrix of the module.

One can always artificially increase the length of a resolution by adding two of the same free modules with an identity map between them. When the ring in question is local we can easily eliminate such artificiality. The ring R is said to be *local* if it has a unique maximal ideal m. We may then speak of a minimal free resolution of a module M; we insist that the image of F_i in F_{i-1} be contained in mF_{i-1}. In this case the length (possibly infinite) and the ranks of the free modules are uniquely determined by M. The length of such a minimal resolution is called the *projective dimension of* M. The only possible difference between two such minimal resolutions comes about from different choices of bases for the free modules. Regular local rings are characterized by the fact that all finitely generated modules over them have *finite* projective dimension. Two themes are easily discernible in the homological conjectures: the first is that modules of finite projective dimension should behave much in the same way that modules over regular local rings behave, and the second is that, in general, many of the homological properties of arbitrary noetherian local rings should be the same as those of Cohen-Macaulay local rings. Another theme that motivated several of the homological conjectures is the question of which ideals can be the support of a module having finite projective dimension. A solution to this question has been elusive. It remains as mysterious as when it was first formulated.

In some sense, the structure and properties of minimal free resolutions should lie at the heart of the homological conjectures. However, this attitude has not been borne out in practice. Indeed, it is more the structure of rings, rather than resolutions, that has penetrated the heart of the homological conjectures.

Most of the homological questions either reduce immediately to the case where the ring is local (by the process of localization at prime ideals of the ring) or were only stated in the local case. *Homological Questions in Local Algebra* presents the solutions of several of the homological conjectures in the equicharacteristic case. The work presented in the book is largely from the 1970s. The author views Serre's classical notes [Se], given in a course at the Collège de France in 1957/58, as the springboard for his own book.

A partial list of the homological conjectures include (1) the New Intersection Conjecture, (2) the Homological Height Conjecture, (3) Bass's Conjecture, (4) the Zerodivisor Conjecture, (5) the Syzygy Conjecture, (6) the Direct Summand Conjecture, (7) the Monomial Conjecture, (8) the existence of Big Cohen-Macaulay modules, (9) the Canonical Element Conjecture, Serre's (10) Vanishing and (11) Nonvanishing of Multiplicities Conjectures, (12) the Horrocks-Buchsbaum-Eisenbud Ranks of Syzygies Conjecture, (13) the Grade (or Strong Intersection) Conjecture, and (14) the Rigidity Conjecture of Auslander. There are numerous interrelations among these conjectures. (For precise formulations of these conjectures, see [Au, BE, EG, Ho1, Ho2, PS1, Ro1, Se].)

Of the above conjectures, (1)–(10) are now theorems in the *equicharacteristic case*, i.e., in the case where the rings in question contain fields. If the ring does not contain a field, we speak of the *mixed characteristic case*. Due to recent work of Paul Roberts, (1)–(4) and (10) are now theorems in full generality (Gillet and Soulé independently did (10)). Conjectures (11)–(14) remain open in almost all cases.

Two fundamental monographs were written in the early 1970s. The first, by Peskine and Szpiro in 1973 [PS1], was a tour de force that solved many of the homological questions for local rings essentially of finite type over fields, for rings of characteristic p, and for other important cases. Peskine and Szpiro pioneered the technique of reduction to characteristic p. One of the strengths of Strooker's book is a nice treatment in Chapter 12, written by van den Dries, of the Artin Approximation Theorem in equal characteristic, and how it can be used. In *Homological Questions in Local Algebra* the general treatment of the homological conjectures and how the Artin Approximation theorem can be used follows the work of the second seminal paper of the early 70s, namely, the monograph of Melvin Hochster in 1975 [Ho1]. Stooker's book allows easier access to this material than was previously available.

Hochster removed all but one of the conditions on the local rings assumed by Peskine and Szprio. He needed to assume that the local rings contained a field. Hochster showed how many (but not all) of the homological conjectures followed from the existence of a so-called big Cohen-Macaulay module. He proved the existence of such a module in characteristic p, then used the Artin Approximation theorem to deduce their existence in equicharacteristic 0 as well. By and large this path is taken by the book under review. Stooker shows the existence of such modules mainly following Hochster's original proofs and presents in some detail how the existence of such a module implies many of the homological conjectures.

The first nine chapters of *Homological Questions in Commutative Algebra* present standard (although advanced) topics in commutative noetherian ring theory. Basics of homology, injective resolutions, dimension, Matlis duality, local cohomology, depth, Cohen-Macaulay modules, Buchsbaum and Eisenbud's criterion for the exactness of a complex, and local duality are covered. The pace is leisurely and kind to the reader. Although some idiosyncrasies can be observed, there are many nice touches, including a section concerning the annihilation of local cohomology, which is used in the proof of the existence of big Cohen-Macaulay modules presented in the book.

One of the basic homological conjectures treated in the later chapters of this book is the New Intersection Conjecture. In characteristic p, and for algebras essentially of finite type over fields, Peskine and Szpiro [PS2] and independently Paul Roberts [Ro1] proved this conjecture. Hochster solved it for all rings that contain fields. Recently, Paul Roberts [Ro3] proved this conjecture in full generality, i.e., he has shown it in the mixed characteristic case. I would like to discuss this conjecture, partly because it provides an easy vehicle to illustrate how the Frobenius endomorphism can be used, as well as how the homological conjectures relate to other themes in commutative algebra. To state the conjecture we must first explain two concepts.

Definition 1.2. Let R be a commutative noetherian ring. The *dimension* (or *Krull dimension*) of R is $\sup\{n$: there exists a chain of prime ideals in R of length $n\}$ and is denoted $\dim(R)$. If P is a prime ideal, the height of P, denoted $\mathrm{ht}(P)$, is $\dim(R_P)$.

For example, fields are of dimension 0; the only chain of prime ideals begins and ends with (0). Dedekind domains have dimension 1, and polynomial rings in n-variables over an m-dimensional noetherian ring have dimension $m+n$.

Definition 1.3. Let R be a local ring with (unique) maximal ideal m and residue field $k = R/m$. If M is a finitely generated R-module supported only at the maximal ideal m (meaning $M_p = 0$ for all primes $p \neq m$) then the *length of* M, denoted $\mathrm{l}(M)$, is the number of copies of simple R-modules (necessarily isomorphic to k) in a saturated filtration of M. This number is independent of the filtration. In case R contains a copy of its residue field (meaning R contains a field k that maps isomorphically onto R/m under the natural homomorphism $R \to R/m$), the length of M is simply the vector space dimension of M over k.

The New Intersection Conjecture states the following:

New Intersection Conjecture. *Let R be a noetherian local ring with maximal ideal m. Let $\mathbf{F}$ be a finite free complex as in* (1.1) *above. If the length of the homology of $\mathbf{F}$ is finite* (*equivalently the support of the homology is* $\{m\}$) *and* $H_0(\mathbf{F}) \neq 0$, *then the length n of $\mathbf{F}$ is at least the dimension of the ring.*

Why should this conjecture be true? The point is that modules of finite length are quite complicated, in fact, the *most* complicated of all finitely generated R-modules. To unwind their structure should require at least $\dim(R)$ steps. Where did the conjecture come from? It is a deluxe version of the dimension inequality for subvarieties of a nonsingular variety, which basically states that the dimension of the intersection of two varieties is at least the sum of their dimensions minus the dimension of the ambient space. One way to obtain complexes as in the statement of the New Intersection Conjecture is to tensor the resolutions of two finitely generated modules having finite projective dimension. If the intersection of the supports of these modules is only the unique closed point (i.e., the unique maximal ideal) then through this process one obtains a complex as in the statement of the conjecture. There are other ways to obtain such complexes where the 0th homology has support that can be thought of as the intersection of two subvarieties. Hence the word "Intersection" in the conjecture.

The New Intersection Conjecture says something about the complexity of the structure of finite length modules. One measure of such complexity is provided by the classical Hilbert-Samuel polynomial. If I is an ideal of a local ring R with maximal ideal m such that the support of R/I is just the maximal ideal m (equivalently, there is a constant k such that m^k is contained in I), then

$\mathrm{l}(R/I^n)$ is, for large n, a polynomial in n of degree equal to $\dim(R)$. Thus, in some sense, ideals I such that R/I has finite length are quite complex; their rate of growth is polynomial of degree equal to the dimension of the ring. A natural thought is to attempt to prove the New Intersection Conjecture by taking advantage of the Hilbert-Samuel polynomial. This actually works in a special case of the conjecture, which we describe below.

The simplest case of the conjecture was proved by Krull [K] more than a quarter of a century before the advent of homological algebra and is known as the Krull (generalized) principal ideal theorem. His theorem states:

Krull's Theorem. *Let R be a noetherian commutative ring. If an ideal I is generated by n elements, $x_1, \ldots, x_n$, then any minimal prime P over I has height at most n.*

One can quickly deduce Krull's theorem from the New Intersection Theorem. To do so requires the introduction of the most important complex in commutative algebra, the Koszul complex. The Koszul complex $\mathbf{K}$ on elements $x_1, \ldots, x_n$ can be defined as follows: Let F be a free R-module of rank n with basis $\{e_i\}$. The Koszul complex is the exterior algebra of F, with a differential of degree -1 uniquely determined by the formulas $d(e_i) = x_i$ and $d(uv) = ((-1)^{\deg u})u \wedge d(v) + d(u) \wedge v$ for homogeneous elements u and v in the exterior power $\bigwedge F$. The resulting complex has length n and $H_0(\mathbf{K}) = R/I$, where I is the ideal generated by the x's. Let P be a minimal prime over I. After localizing at P we obtain a local ring of dimension $\mathrm{ht}(P)$, and the support of the homology of $\mathbf{K}$ (after localization) is $\{PR_P\}$. Moreover, the 0th homology is not zero. The New Intersection Theorem tells us that the length of the complex, namely, n, is at least $\mathrm{ht}(P)$, i.e., $\mathrm{ht}(P) \leq n$, which is Krull's theorem.

The key tool used in the proof of the homological conjectures (in all but one case) is the Frobenius endomorphism in characteristic p. If R has characteristic $p > 0$ and $\mathbf{F}$ is a finite free complex as in (1.1), then we may choose bases of the free modules and represent the maps in the resolution as matrices with entries in R. Raising these entries to the qth power, where q is a power of p, preserves the fact that $\mathbf{F}$ is a complex and does not change the support of the homology. When $q = p$ we refer to this process as *applying Frobenius to* $\mathbf{F}$. It is important to emphasize that applying the Frobenius endomorphism preserves the fact that $\mathbf{F}$ is a complex. No similar process is known in characteristic 0. That this fact is valid in characteristic p is due to the formula $(a+b)^p = a^p + b^p$, which holds for elements a and b in a ring of characteristic p. One obtains a contradiction from a counterexample $\mathbf{F}$ of the New Intersection Conjecture by applying Frobenius to the counterexample $\mathbf{F}$ repeatedly. What was once improbable becomes absurd after enough iterations.

There is no similar process known in general in characteristic 0 or mixed characteristic. Why should applying the Frobenius help? One way to see why it might is the case of Krull and the Koszul complex. If I is generated by $x_1, \ldots, x_n$ and is supported only at the maximal ideal m of a local ring R, then one can show that $\mathrm{l}(R/(x_1^q, \ldots, x_n^q))$ is at most Cq^n, where C is a positive constant. On the other hand, $(x_1^q, \ldots, x_n^q)$ will be contained in m^q and so this length will be at least $\mathrm{l}(R/m^q)$, which in turn is, for large q, a polynomial of degree $\dim(R)$ in q. Hence $n \geq \dim(R)$, as we claimed.

Note that we did not really need the Frobenius in this case; q could be any positive integer, not only a power of p. This is due to a special property of Koszul complexes. Raising the entries to the qth power preserves the fact that it is a complex. This will not work for any other complex I know of in characteristic 0, but illustrates how obtaining a new complex by raising the entries of the matrices to powers can allow one to bring into play the classical Hilbert-Samuel polynomial.

One of the mildly unfortunate aspects of our understanding of the homological conjectures has been the relative isolation of them from other important themes in commutative algebra, e.g., resolution of singularities, valuation theory, blowing up of ideals, Jacobians, differentials, primary decomposition, and various closure theories for ideals (such as the integral closure). There are notable exceptions to this assertion. Paul Roberts [Ro2] used resolution of singularities to give a new proof of the New Intersection Conjecture for local rings essentially of finite type over the complex numbers. A better synthesis between classical ideal theory and homological algebra would be welcome. In fact, some of the conjectures are an attempt to bridge this gap. Fortunately, there has been progress since the 1970s.

Since 1980 there has been a great deal of significant work done on the homological conjectures. The introduction of *tight closure* [HH1] has given a better understanding of the homological conjectures in the equicharacteristic case, not only proving stronger results, but also providing a clearer understanding of principles behind these conjectures. Several other results, not homological in appearance, have been shown to follow from the same basic themes. In particular, tight closure has been able to replace analytic methods in many instances.

Earlier work in the 1970s also tried to give more explicit proofs of some of the homological conjectures. The "structure theorems" of Buchsbaum and Eisenbud from the early 1970s can be thought of as providing another attempt at a more concrete realization of some of the homological conjectures. Recent work suggests there is much more to do along these lines. Robert's work in mixed characteristic needs to be studied and interpreted for a general audience. His use of resolution of singularities to give a proof of the New Intersection Theorem is surely the tip of an iceberg. The work of Evans and Griffith [EG] in the early 1980s related homological methods to ideal theory and vector bundles. The recent proof in [HH2] that the integral closure of a complete local domain of characteristic p in an algebraic closure of its fraction field is a big Cohen-Macaulay algebra has many ramifications. Strooker's book provides a helpful introduction to the homological conjectures, but the scope of *Homological Questions in Local Algebra* did not allow the author to present much of the recent work. One might hope for a future book detailing much of the new work that has been done.

Bibliography

[Au] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. **5** (1961), 631–647.

[AB] M. Auslander and D. Buchsbaum, *Unique factorization in regular local rings*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 733–734.

[Ar] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–56.

[BE] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), 447–485.

[EG] G. Evans and P. Griffith, *The syzygy problem*, Ann. of Math. (2) **114** (1981), 323–333.

[Ho1] M. Hochster, *Topics in the homological theory of modules over commutative rings*, C.B.M.S. Regional Conf. Ser. in Math., vol. 24, Amer. Math. Soc., Providence, RI, 1975.

[Ho2] ——, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. Algebra **84** (1983), 503–553.

[HH1] M. Hochster and C. Huneke, *Tight closure, invariant theory and the Briancon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.

[HH2] ——, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. of Math. (2) (to appear).

[K] W. Krull, *Primidealketten in allgemeinen Ringbereichen*, Heidelberger Akad. Wiss. Math.-Natur. Kl. **7** (1928), 7.

[PS1] C. Peskine, and L. Szpiro, *Dimension projective finie et cohomologie locale*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 323–359.

[PS2] ——, *Syzygies et multiplicités*, C. R. Acad. Sci. Paris (A) **278** (1974), 1421–1424.

[Ro1] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 103–106.

[Ro2] ——, *Cohen-Macaulay complexes and an analytic proof of the new intersection conjecture*, J. Algebra **66** (1980), 220–225.

[Ro3] ——, *Le theoreme d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 177–180.

[Se] J.-P. Serre, *Algebre locale-multiplicités*. Lecture Notes in Math., vol. 11, Springer-Verlag, New York and Berlin, 1965.

CRAIG HUNEKE
PURDUE UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 2, April 1992

0273-0979/92 $1.00 + $.25 per page

Geometry of Banach spaces, duality mappings and nonlinear problems by Ioana Cioranescu, Kluwer Academic Publishers, Dordrecht, 1990, 260 pp., $99.00. ISBN 0-7923-0910-3.

What can replace the inner product in non-Hilbert Banach spaces? To answer this natural question we may proceed as follows. Let X^* be the dual space of a real Banach space X, and denote the norms of both X and X^* by $|\cdot|$. For each x in X define

$$J(x) = \{x^* \in X^* \colon (x, x^*) = |x|^2 = |x^*|^2\}.$$

This weak-star compact convex subset of X^* is always nonempty by the Hahn-Banach theorem, and the mapping $J \colon X \to 2^{X^*}$ is called the normalized duality mapping of X. For x and y in X we now define two semi-inner-products by

$$(y, x)_+ = \max\{(y, x^*) \colon x^* \in J(x)\}$$

and

$$(y, x)_- = \min\{(y, x^*) \colon x^* \in J(x)\}.$$

Equivalently,

$$(y, x)_+ = \lim_{t \to 0+} (|x + ty|^2 - |x|^2)/(2t)$$