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Vector Lyapunov functions and stability analysis of nonlinear systems, by V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram. Kluwer Academic Publishers, Dordrecht, 1991, 172 pp., $79.00. ISBN 0-7923-1152-3

1. INTRODUCTION

One hundred years ago a Russian mathematician, A. M. Lyapunov, published a major work (printed or translated variously as [10, 11]) setting forth a method for studying stability properties of solutions of ordinary differential equations. The method is based on the construction of a function (now called a Lyapunov function) that serves as a generalized norm of a solution. Its appeal comes from the fact that properties of the solutions are derived directly from the differential equation itself (whence comes the name "Lyapunov's direct method").

This method is recognized by many investigators as the only general way of dealing effectively with stability questions of nonlinear ordinary differential equations. But for the past forty years it has also been used with marked success in the study of functional and partial differential equations. A careful look at many of these results shows that the method introduces a unifying thread through these historically diverse areas. And this is a thread that seems well worth studying.

2. Stability

Let D be an open set in R^n with $0 \in D$ and let $f\colon [0, \infty) \times D \to R^n$ be continuous. Then

$$x' = f(t, x) \quad (' \text{ denotes } d/dt) \tag{1}$$

is a system of ordinary differential equations and if $(t_0, x_0) \in [0, \infty) \times D$ then there is a solution $x(t, t_0, x_0)$ satisfying (1) on an interval $t_0 \le t < \alpha$ with $x(t_0, t_0, x_0) = x_0$; if the solution remains in a compact subset of D then $\alpha = \infty$. This is an existence theorem and it does not produce a solution. The insurmountable problem we face in finding a solution is that we must integrate an unknown function.

One alternative is to study qualitative properties of solutions from information contained in the differential equation itself. In many problems of interest there are special solutions that are readily found by inspection, called equilibrium or constant solutions. It is then fruitful to study the behavior of solutions starting near the constant solution, and this is stability theory. There is a vast number of important types of stability, but two of the most fundamental ones are called uniform stability and uniform asymptotic stability. The typical example is the model of a pendulum. When it hangs straight down or stands straight up, it is in equilibrium. When the equilibrium is disturbed, the pendulum may move far from that equilibrium (instability), it may oscillate periodically and stay near that equilibrium (stability), or it may oscillate and approach that equilibrium (asymptotic stability).

Definition 1. Let $f(t, 0) = 0$. The solution $x(t) = 0$ of (1) is

(i) *uniformly stable* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \ge 0,\ t \ge t_0,\ |x_0| < \delta]$ imply that $|x(t, t_0, x_0)| < \varepsilon$;

(ii) *uniformly asymptotically stable* if it is uniformly stable and if there is an $\eta > 0$ and for each $\mu > 0$ there is a $K > 0$ such that $[t_0 \ge 0, |x_0| < \eta,\ t \ge t_0 + K]$ imply that $|x(t, t_0, x_0)| < \mu$.

Lyapunov's direct method enters in the following way. For a given scalar function $V\colon [0, \infty) \times D \to [0, \infty)$, if (t_0, x_0) is given then there is a solution $x(t) = x(t, t_0, x_0)$ and, even if V bears no relation to (1), still $V(t, x(t))$ is a well-defined function of t so long as the solution remains in D. If V has continuous first partials, then we may *compute*

$$\begin{aligned}(d/dt)V(t, x(t)) \overset{\text{def}}{=} V'_{(1)}(t, x) &= \operatorname{grad} V \cdot f + (\partial V/\partial t) \\ &= \sum_{i=1}^{n} (\partial V/\partial x_i) f_i(t, x) + (\partial V/\partial t)\end{aligned} \tag{2}$$

by the chain rule. Here, f_i is the ith component of f. If $V'_{(1)}(t, x) \le 0$, then V is called a Lyapunov function for (1). Thus, the derivative of $V(t, x(t))$ is a known function of (t, x) obtained directly from (1) itself.

Lyapunov's idea was that if V is positive definite and chosen so shrewdly that $V'_{(1)}(t, x) \le 0$ then $V(t, x(t)) \le V(t_0, x_0)$, which may yield important boundedness properties of $x(t)$ itself. Theorem 1 is the fundamental classical result for (1). Here, a wedge is a continuous strictly increasing function $W: [0, \infty) \to [0, \infty)$ with $W(0) = 0$. All functions W_i are wedges in this discussion.

Theorem 1. *Suppose there is a differentiable function* $V: [0, \infty) \times D \to [0, \infty)$ *and wedges* W_i.

(i) *If* $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$ *and* $V'_{(1)}(t, x) \leq 0$, *then the zero solution of* (1) *is uniformly stable.*

(ii) *If* $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$ *and* $V'_{(1)}(t, x) \leq -W_3(|x|)$, *then* $x = 0$ *is uniformly asymptotically stable.*

The contents of Theorem 1 are not idle suppositions. If f is smooth enough then the theorem can be reversed; stability can be characterized by Lyapunov functions.

The reviewed book devotes the first fifty-two pages to an introduction, applications, and refinements of these concepts. While Theorem 1 can be reversed, it is an existence theorem and seldom do we find a Lyapunov function satisfying its conditions perfectly. There is a long and interesting line of investigation that proceeds from imperfect fulfillment of the conditions of Theorem 1. Its genesis is traced to Marachkov [13] in 1940 and it has had much impact on the development of functional differential equations.

Theorem 2. *Suppose that* $V: [0, \infty) \times D \to [0, \infty)$ *with* $V(t, 0) = 0$, $W_1(|x|) \leq V(t, x)$, $V'_{(1)}(t, x) \leq -W_3(|x|)$, *and suppose that* $f(t, x)$ *is bounded for* $|x|$ *bounded. Then* $x = 0$ *is asymptotically stable.*

The conclusion is similar, but weaker, than uniform asymptotic stability. The loss of the upper wedge on V is replaced by $f(t, x)$ being bounded.

Examining this result, Krasovskiĭ [7, p. 67] had an idea that started a small industry. For Krasovskiĭ's theorem we note that a set is said to be positively invariant if any solution entering the set remains in it for all future time. Moreover, observe that if $f(t, x)$ is periodic in t then it is bounded for x bounded.

Theorem 3. *Let* $f(t, x)$ *be periodic in* t, *and suppose that* $V: [0, \infty) \times D \to [0, \infty)$ *with* $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$ *and* $V'_{(1)}(t, x) \leq 0$. *If the only positively invariant set in which* $V' = 0$ *is* $x = 0$, *then* $x = 0$ *is asymptotically stable.*

Yoshizawa [14] extended this result to nonperiodic systems and concluded that bounded solutions remaining in D approach the set where $V' = 0$ so long as f is bounded for x bounded. LaSalle [9] concluded that bounded solutions approach the largest invariant set where $V' = 0$. Hale [4] and Haddock and Terjéki [2] extend the idea to functional differential equations, while Henry [6, p. 91] has a nice formulation for partial differential equations.

One of the most interesting consequences of Marachkov's result is the adverse manner in which it affected functional differential equations for more than 20 years.

Let $h > 0$ and let C be the Banach space of continuous functions $\varphi: [-h, 0] \to R^n$ with the supremum norm $\|\cdot\|$. For a continuous function $x: [-h, A) \to R^n$ with $A > 0$, we denote by x_t the restriction of x to the interval $[t-h, t]$

translated back to $[-h, 0]$ so that $x_t \in C$ and $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$. Let C_H be the H-ball in C and $F: [0, \infty) \times C_H \to R^n$ be continuous, take bounded sets into bounded sets, and let $F(t, 0) = 0$. Then

$$x'(t) = F(t, x_t) \tag{3}$$

is a functional differential equation with finite delay and it has the zero solution. For a given $t_0 \geq 0$ and $\varphi \in C_H$, there is at least one solution $x(t, t_0, \varphi)$ of (3) on an interval $[t_0, \alpha)$ with $x_{t_0} = \varphi$; if it remains in C_K with $K < H$ then $\alpha = \infty$. Stability definitions for (3) are obtained from those for (1) by replacing x_0 by φ and $|\cdot|$ by $\|\cdot\|$.

About 1957 Krasovskiĭ [7, pp. 126–175] developed stability theory for (3) in a marvelously simple way. In Theorem 1 he essentially replaced x_0 by φ and $|\cdot|$ by $\|\cdot\|$. The result was true, easy to prove, and it even had a converse. Unfortunately, it was almost thirty years before investigators learned how to construct Lyapunov functions of that type and Krasovskiĭ abandoned it (cf. [7, p. 151]) in favor of a version patterned after Marachkov's result that was crippled by the requirement that $F(t, \varphi)$ be bounded for φ bounded. Examples were readily found for the latter formulation and this result remained the standard (cf. [5, p. 105]) until the following result [1] appeared in 1978. Here, $|||\cdot|||$ denotes the L^2-norm.

Theorem 4. *Let* $V: [0, \infty) \times C_H \to [0, \infty)$ *be differentiable with* $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi(0)|) + W_3(|||\varphi|||)$ *and* $V'_{(3)}(t, x_t) \leq -W_4(|x(t)|)$. *Then the zero solution of* (3) *is uniformly asymptotically stable.*

Investigators have believed that the norm in the upper wedge might be replaced by the supremum norm, but a recent paper by a young Hungarian, Makay [12], indicates that this is likely to be false.

This is a small sample of the traditional presentation as may be found in the standard texts of Hahn [3], Krasovskiĭ [7], and Yoshizawa [15]. And much of this is also found in the reviewed book. But the authors focus on two additional ideas that were mainly developed by Lakshmikantham and Leela [8]. To obtain uniform asymptotic stability, they ask for one Lyapunov function satisfying the conditions of Theorem 1(i), thus ensuring uniform stability. In addition, they ask for another Lyapunov function that is bounded and has a positive definite derivative (It could just as well be negative definite.) Clearly, this is more flexible than asking for both conditions (i) and (ii) of Theorem 1 for the same Lyapunov function. Taking this to its logical conclusion, they ask for a vector Lyapunov function. Next, note that Theorem 1(ii) implies that $V'_{(1)}(t, x) \leq -W_4(V(t, x))$. Taking this also to its logical conclusion, they ask for a set of differential inequalities $V'(t, x) \leq g(t, V(t, x))$, where $r' = g(t, r)$ has certain stability properties. They conclude that $V(t, x(t)) \leq r(t)$, where $r(t)$ is a certain solution of $r' = g(t, r)$. (Here, V, G, and r are vectors. The inequalities are componentwise.)

Beyond any doubt, their treatment is more general than the standard ones.

The reviewed book also has significant material on boundedness that has very interesting application to fixed point theory and to the existence of periodic solutions. It contains general treatment of functional differential equations with infinite delay, Volterra equations, control theory, partial differential equations, difference equations, and applied problems. It is the work of investigators who have spent thirty fruitful years on the subject and it is a welcome contribution to the literature.

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Walsh series and transforms, by Golubov, Efimov, and Skvortsov. Kluwer Academic Publishers, Dordrecht, The Netherlands 1991, 367 pp., $169.00. ISBN 0-7923-1100-0

An *orthonormal system* on the interval $[0, 1)$ is a sequence of functions $\phi_0, \phi_1, \dots$ that satisfies

$$\int_0^1 \phi_k(x)\phi_j(x)\,dx = \begin{cases} 1 & k = j, \\ 0 & k \neq j. \end{cases}$$

The Walsh system occupies a unique position among orthonormal systems on $[0, 1)$. It is the completion of the Rademacher system, a prototype for all