idea since the geometry of the associated functionals is more transparent to analyze in these cases. Thus, in Chapter 1, solutions can be obtained as strict minimizers of an appropriate energy functional, and in later chapters, the nonlinear eigenvalue problems associated with periodic solutions are more easily described.

One difficulty with the book is the strange and interesting ambiguity of Hamiltonian systems reacting with critical point theory. Thus, physical ideas of integrable Hamiltonian systems, conservation laws, quasi-periodic motions, and the K.A.M. theorem are hardly mentioned even though these concepts are very fundamental for the current study of nonlinear Hamiltonian systems; while for Critical Point Theory the ideas described seem relatively unmotivated instead of coming from a coherent geometric picture of mechanics. Thus, the book focuses on finite-dimensional Hamiltonian systems and does not discuss infinite-dimensional systems at all, even though such an extension is very important in contemporary research. The book focuses on Hamiltonian systems connected with the theory of oscillations in engineering, but the ideas connected with periodic motions in celestial mechanics are hardly mentioned, even though they both are very basic and very classic. One exception occurs in Chapter 9.2, on p. 207, when periodic solutions of autonomous second-order systems near equilibrium are considered. But, the associated classical result associated with the Russian mathematician, Liapunov, is not described. Nor is the historical context of why this result is so important. The idea of periodic motions associated with Kepler's two body problem are not described, nor the context for the historic search for periodic motions in the more general three-bodies case.

All in all, with the above caveats, the book of Mawhin and Willem is a very fine introduction to the ideas of utilizing contemporary functional analysis for the classical physical problem of periodic motions of finite-dimensional Hamiltonian systems. Many other books on this topic are being written, and it is hoped that the volumes will not only popularize the subject but lead to a great flowering of this discipline utilizing the most contemporary results in mathematical analysis and topoplogy.

M. S. Berger
University of Massachusetts, Amherst

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 2, April 1992

0273-0979/92 $1.00 + $.25 per page

Capacity theory on algebraic curves, by Robert S. Rumely, Lecture Notes in Mathematics, vol. 1378, Springer-Verlag, Berlin, Heidelberg, and New York, 1989, 437 pp., $37.50. ISBN 3-540-51410-4

If you asked a number theorist, an analyst, a logician, and an arithmetic geometer to define capacity theory, you would be likely to hear four different answers. To a number theorist, capacity theory has to do with the possible locations of sets of Galois conjugate algebraic numbers, as well as with criteria for power series to be rational functions. An analyst might discuss potential theory and extremal subharmonic functions. Logicians have found capacity theory to be a powerful tool in studying the decidability of diophantine problems. Finally, an arithmetic geometer might view capacity theory in terms of arithmetic intersection theory, though the precise connection between the two is an interesting problem at present.

Robert Rumely's book, *Capacity theory on algebraic curves*, is a remarkably thorough and complete description of an arithmetic capacity theory on curves over global fields. To develop this theory, Rumely gives among other things a generalization to Riemann surfaces of classical capacity theory on $\mathbf{C}$ and a new analysis of Neron's local height pairing on curves over nonarchimedean local fields. I will describe Rumely's main number theoretic result below. One consequence of it is that Hilbert's tenth problem concerning decision procedures for systems of diophantine equations has an affirmative solution over the ring of all algebraic integers. Thus Rumely's book has something specific to offer to each of the four groups of mathematicians mentioned above. For all readers it provides a fine example of how number theory, analysis, and arithmetic geometry can join forces.

At 437 pages, Rumely's book is not short. While it contains lucid summaries of the main ideas, the bulk of the text is concerned with setting the right foundations. Before reading it I would strongly suggest reading the brief summary given in [8]. In this review I will focus on Rumely's main result and I will try to point the reader to some useful mileposts in his book. At the end I will discuss recent related results and some personal impressions about the subject and this book.

The inspiration for Rumely's main result is the following theorem of Fekete and Szegö. Suppose E is a compact subset of $\mathbf{C}$ that is stable under complex conjugation. One definition of the capacity of E is $\gamma(E) = e^{-V(E)}$ where

$$V(E) = \inf_{\nu(E)=1} \iint_{E\times E} -\log|x-y|\, d\nu(x)\, d\nu(y) \tag{1}$$

and the inf is taken over all probability measures ν on E. Here $V(E)$ is just the minimal energy of a charge distribution of total mass 1 on E with respect to the logarithmic potential $-\log|x-y|$. The unique probability measure ν_E for which the minimum is attained is called the equilibrium distribution of E.

Theorem 1 (Fekete and Szegö [4, 5]). (a) *If $\gamma(E) < 1$ then there is an open neighborhood U of E containing only finitely many complete sets of conjugates of algebraic integers, and*

(b) *If $\gamma(E) > 1$ then every open neighborhood U of E contains infinitely many such sets.*

Building on work of Cantor [1] for the projective line, Rumely proves an adelic generalization of this theorem for arbitrary curves $\mathscr{C}$ over a global field K. The set E is replaced by a pair $(\mathscr{E}, \mathscr{X})$ consisting of a product $\mathscr{E} = \prod_v E_v$ over the places of K of subsets E_v of $\mathscr{C}(\overline{K}_v)$ together with a finite $\mathrm{Gal}(\overline{K}/K)$-stable subset $\mathscr{X}$ of $\mathscr{C}(\overline{K})$. Here $\overline{K}$ is an algebraic closure of K contained in an algebraic closure $\overline{K}_v$ of the completion K_v of K at v. If $\mathscr{E}$ and $\mathscr{X}$ satisfy suitable hypotheses, Rumely defines a capacity $\gamma(\mathscr{E}, \mathscr{X}) \geq 0$ via potential theory, and proves the following theorem.

Theorem 2 (Rumely). *Suppose that each E_v is stable under* $\mathrm{Gal}(\overline{K}_v/K_v)$ *and that $\gamma(\mathscr{E}, \mathscr{X})$ is well defined.*

(a) *If* $\gamma(\mathscr{E}, \mathscr{X}) < 1$, *there exists a product* $\mathscr{U} = \prod_v U_v$ *of open neighborhoods* U_v *of* E_v *in* $\mathscr{E}(\overline{K}_v)$ *that contains only finitely many complete sets of* $\mathrm{Gal}(\overline{K}/K)$*-conjugate points of* $\mathscr{E}(\overline{K})$, *and*

(b) *If* $\gamma(\mathscr{E}, \mathscr{X}) > 1$ *then all* $\mathscr{U}$ *as above contain infinitely many such sets of conjugate global points.*

What is remarkable about this theorem is that it is simultaneously general, quantitative, and effective. For example, the theorem is general enough to imply (with Bertini's Theorem) that any absolutely irreducible affine variety $\mathscr{V}$ over K has a point over the ring $\overline{O}$ of all algebraic integers if and only if $\mathscr{V}$ has a point over the integers $\overline{O}_v$ of $\overline{K}_v$ for all nonarchimedean places v of K. At the other extreme, Theorem 2 has many concrete applications to the existence of units in number fields and integral points on elliptic curves whose conjugates satisfy various congruence conditions and archimedean constraints. For a sample of these latter results, see pp. 4–6, 368–372.

I will now discuss the chapters of Rumely's book.

Chapter 1 is introductory. After reading this, I suggest reading the exposition of the proof of Theorem 1 given in §1 of Chapter 6. This will greatly clarify the basic principle of the proofs. Potential theory enters into Theorem 1 in the following way. Let $h(z) \in \mathbf{R}[z]$ be a monic polynomial of large degree whose zeroes are distributed throughout E according to a discrete approximation to the equilibrium distribution ν_E defined after equation (1). Then $\frac{1}{\deg(h(z))} \log|h(z)|$ is approximately equal to

$$(2) \qquad \int_E \log|z-w|\, d\nu_E(w) = -u_E(z) = G(z, \infty; E) - V(E)$$

for z away from E, where $u_E(z)$ and $G(z, \infty; E)$ are called the conductor potential and the Green's function of E, respectively. By adjusting the coefficients of $h(z)$ to be integral we arrive at a monic auxillary polynomial $f(z) \in \mathbf{Z}[z]$ that can be used to control the algebraic integers appearing in Theorem 1. In a similar way, the basis for Theorem 2 is the construction of auxillary rational functions $f(z) \in K(\mathscr{E})$ having the local properties discussed in §2 of Chapter 6. The basic problem is to build a potential theory, first for each place v of K and then adelically, which is sufficient to construct such $f(z)$.

Rumely's first step in developing v-adic potential theory is to replace the classical potential $-\log|x-y|$ by $-\log[x, y]_\zeta$, where $[x, y]_\zeta$ is a "canonical distance function" between points $x, y \in \mathscr{E}(\overline{K}_v)$ normalized so that $\zeta \in \mathscr{E}(\overline{K}_v)$ is the "point at infinity." In §2 he gives several constructions of $[x, y]_\zeta$ and shows that it may be recovered from a two variable "Arakelov function" $((x, y))_v$. For archimedean v, $((x, y))_v$ is (up to normalization) an Arakelov Green's function in the usual sense, and depends on the choice of a volume form on $\mathscr{E}(\overline{K}_v)$. Using electric circuit theory and a detailed analysis of Neron's local height pairing, Rumely develops a remarkable counterpart of Arakelov Green's functions for nonarchimedean v. In this case, $((x, y))_v$ depends on the choice of a probability measure on a "reduction graph" constructed from the special fibres of "well-adjusted" integral models of $\mathscr{E}$ over sufficiently large finite extension of K_v.

In §§3 and 4 Rumely treats local capacity theory. Depending on the choice of $\zeta \in \mathscr{C}(\overline{K}_v)$, he defines a local capacity $V_\zeta(E_v)$, an equilibrium measure on E_v, a conductor potential $u_{E_v}(z, \zeta)$, and a Green's function $G(z, \zeta; E_v)$ analogous to the classical case. If v is nonarchimedean the theory applies to many interesting E_v that are not compact, and the analysis of which E_v are "algebraically capacitable" is subtle. For all v, Rumely considers when there are rational functions $h_v(z) \in K_v(\mathscr{C})$ for which $\frac{1}{\deg(h_v)} \log |h_v(z)|_v$ is close to a prescribed linear combination of Green's functions for z in a specified domain. Such "approximation theorems" generalize those involved in Theorem 1. Rumely also shows his local capacity $V_\zeta(E_v)$ for reasonable E_v coincides with the natural counterparts of the transfinite diameter and Chebyshev constant of a compact subset of $\mathbf{C}$.

Following Cantor [1], Rumely defines in §5 the global capacity $\gamma(\mathscr{E}, \mathscr{X})$ to be the value of a certain matrix game constructed via the local Green's functions $G(z, \zeta; E_v)$. This definition may seem bizarre at first, but it arises naturally from viewing the construction of the auxillary functions $f(z) \in K(\mathscr{C})$ as an optimization problem. The proof of Theorem 2 is carried out in §6. The most delicate part of the proof is the "patching construction," which generalizes the passage from real to integral auxillary polynomials in the proof of Theorem 1.

I will now discuss some recent research related to this book. Rumely's treatment of archimedean Green's functions suggest asking similar questions about the higher dimensional counterparts suggested by Siciak, Zaharjuta, and others (see the Bulletin review [10] by B. A. Taylor). Rumely's approach to nonarchimedean Green's functions, together with Kani's [7], may eventually prove relevant to an important question of Gillet and Soulé's [6, p. 96] concerning how to do arithmetic intersection theory analytically at all places. Motivated by [6], I defined in [2] a "sectional capacity" for varieties of arbitrary dimension. In [9] Rumely establishes the relation between sectional capacity in dimension 1 and his and Cantor's capacity $\gamma(\mathscr{E}, \mathscr{X})$.

My own impression of arithmetic capacity theory is that it still has some way to go before achieving the kind of spectacular applications arithmetic intersection theory has found to the Mordell and Lang conjectures [11, 3]. It is encouraging nonetheless that some of Rumely's ideas play a role in important recent work of Zhang [12] in Arakelov theory. Concerning Rumely's book, I found it to be very carefully written, and it succeeds in its goal of carefully laying foundations. On the other hand, it can be tiring to read, particularly in the technically detailed middle chapters. This may be hard to avoid given the complexity of the material.

Capacity theory on algebraic curves contains a wealth of ideas and techniques, and it has something to offer a wide variety of mathematicians. The book is an attractive combination of analysis, number theory, and arithmetic geometry, and for readers with some endurance it will be every rewarding.

REFERENCES

1. D. Cantor, *On an extension of the definition of transfinite diameter and some applications*, J. Reine Angew. Math. **316** (1980), 160–207.

2. T. Chinburg, *Capacity theory on varieties*, Compositio Math. **80** (1991), 75–84.

3. G. Faltings, *The general case of S. Lang's Conjecture*, preprint, 1991.

4. M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten*, Math. Z. **17** (1923), 228–249.
5. M. Fekete and G. Szegö, *On algebraic equations with integral coefficients whose roots belong to a given point set*, Math. Z. **63** (1955), 158–172.
6. H. Gillet and C. Soulé, *Arithmetic intersection theory*, Inst. Hautes Études Sci. Publ. Math. **72** (1990), 93–174.
7. E. Kani, *Potential theory on curves*, Proc. Internat. Conf. Number Theory, Quebec, 1987 (to appear).
8. R. Rumely, *Capacity theory on algebraic curves and canonical heights*, No. 22 Fasc. 2, Groupe d'étude d'Analyse ultramétrique (Y. Amice, G. Christol, P. Robba, eds.) 12^e Année, 1984/1985, Paris.
9. ——, *On the relation between Cantor's capacity and Chinburg's sectional capacity*, preprint (1990).
10. B. A. Taylor, book review of *Capacities in complex analysis*, by U. Cegrell, Bull. Amer. Math. Soc. (N.S.) **24** (1991), 213–216.
11. P. Vojta, *Siegel's theorem in the compact case*, Annals of Math. **133** (1991), 509–548.
12. S. Zhang, *Positive line bundles on arithmetic surfaces*, Columbia Univ. Thesis, 1991.

TED CHINBURG
UNIVERSITY OF PENNSYLVANIA

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 26, Number 2, April 1992

0273-0979/92 $1.00 + $.25 per page

Nonlinear elliptic and evolution problems and their finite element approximations, by A. Ženišek. Academic Press, London 1990, 422 pp., $45.00. ISBN 0-12-779560-X

For a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ consider the boundary value problem

$$(1) \qquad -\sum_{i=1}^{2} \frac{\partial}{\partial x_i} b_i(x, u, \nabla u) + b_0(x, u, \nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where b_i: $\Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, $i = 0, 1, 2$, are smooth functions satisfying the ellipticity and growth conditions

$$(2) \qquad \sum_{i,j=0}^{2} \frac{\partial b_i}{\partial \xi_j}(x, \xi)\eta_i\eta_j \geq \alpha(\eta_1^2 + \eta_2^2) \quad \forall x \in \Omega, \ \forall \xi, \eta \in \mathbb{R}^3;$$

$$(3) \qquad \left|\frac{\partial b_i}{\partial x_j}(x, \xi)\right| + |b_i(x, \xi)| \leq c(1 + |\xi|), \quad \forall x \in \Omega, \ \forall \xi \in \mathbb{R}^3;$$

$$(4) \qquad \left|\frac{\partial b_i}{\partial \xi_j}(x, \xi)\right| \leq c \quad \forall x \in \Omega, \ \xi \in \mathbb{R}^3.$$