BOOK REVIEWS

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Critical point theory and submanifold geometry, by Richard S. Palais and Chuu-lian Terng. Lecture Notes in Math., vol. 1353, Springer-Verlag, Berlin and New York, 270 pp., $26.00. ISBN 3-540-50399-4

Critical point theory and Hamiltonian systems, by Jean Mawhin and Michel Willem. Appl. Math. Sci., vol. 74, Springer-Verlag, Berlin and New York, 275+ pp., $54.00. ISBN 0-387-96908-X

The two books under review relate the Calculus of Variations and two key topics in the mathematical sciences. The first book connects the Calculus of Variations with the Geometry of Submanifolds. The second book relates the Calculus of Variations with periodic motions of finite-dimensional Hamiltonian systems. Before commencing with the review of each of these books, a few words need to be stated about the Calculus of Variations and its relation to Critical Point Theory. I always enjoyed chatting with the late and renowned Marston Morse about the Calculus of Variations and Critical Point Theory. He had a true intellectual passion for the Calculus of Variations and Geometry and loved to tell stories of his adventures with the practitioners of these mathematical arts. One of his main stories pertained to the traditional Calculus of Variations subject matter and proponents of this field. Morse told the story of the difficulties he had in the 1920s convincing the experts at the time to distinguish between critical points that are and are not extrema or absolute minimizers of associated functionals. Indeed, he, himself, used this distinction with great mastery and, as is well known, was able to connect his study of critical points with the topology of both finite- and infinite-dimensional manifolds, a subject now called Morse Theory.

The Calculus of Variations has a long and fascinating history in Mathematics, dating back at least to the Greek, Hero, and continuing on with the studies of Euler and LaGrange, to mention only two of the great practitioners in the post-Calculus era. The outlook for the Calculus of Variations seemed very bright in the nineteenth century with Gauss and Riemann making notable contributions. But then Weierstrass found his famous counterexample to the Dirichlet problem that put the subject in a shady mathematical light for more than a century, with the notable exception of Poincaré. Studies by Hilbert, Weyl, and Courant were crucial by remedying this situation between 1900–1950. Then, in the 1960s, a number of younger researchers realized that Weierstrass' objections could be nicely superseded by utilizing the functional analysis of Hilbert space and the newly developed topology of manifolds with the newly developed theory of regularity for variational problems with linear elliptic Euler-LaGrange equations.

Now, with this preview, let us turn to the books under review.

The first volume under review, by Palais and Terng, is more informal in nature, in keeping with the Lecture Notes Series in Mathematics. The mathematical community is the beneficiary of the happiness of Palais and Terng because their fine mathematical collaboration resulted in this volume. The book is divided in two parts, the first part part dealing in Submanifold theory and the second part dealing with Critical Point Theory, otherwise known by the old-fashioned name of Calculus of Variations in the Large. The two parts are connected by the Morse Index Theorem. The first eight chapters of the book deal with Submanifold theory, dealing with submanifolds of space forms, surfaces of constant mean and Gaussian curvature and their extensions to Weingarten surfaces, the topology of immersed submanifolds in Euclidean space, and focal points with extensions to the Riemannian geometry of Hilbert manifolds. Thus, one chapter in the book, Chapter Five, deals with infinite-dimensional transformation groups focusing mainly on isometric actions of these Hilbert manifolds. Thus we see a key point in the text is the development of an interesting theory of infinite-dimensional Riemannian manifolds utilizing the infinite-dimensional notions of Fredholm Operators and proper maps to use the techniques of differential topology and Morse theory in infinite dimensions. The book has an interesting Chapter Six dealing with submanifolds that are isoparametric, meaning submanifolds of a space form with flat normal bundles and with principal curvatures along any parallel normal vector fields constant. This topic has been greatly extended in recent years, but the extension in this volume to Hilbert manifolds seems quite new. Most of the first five chapters of the book, which are very clearly written, are concerned with finite-dimensional submanifolds; beginning at the very end of Chapter Five and continuing along for the next four chapters, the infinite-dimensional case of Hilbert manifolds takes over top billing. In particular, in Chapter Eight the authors use the Calculus of Variations relating to infinite-dimensional Morse Theory developed in part two to prove that a nondegenerate distance function defined on an isoparametric manifold of Hilbert space is a perfect Morse function.

This last result serves as the motivation for the second part of the book on Critical Point Theory. This part consists of three chapters with the first part being elementary theory, the second part, more advanced theory, and the third part relating to the Calculus of Variations as such. The first chapter in this part, dealing with elementary Critical Point Theory, is very elegantly done. It incorporates the standard results on Hilbert space relative to Minimax principles, steepest descent, deformations, elements of Lusternik-Schnirelman critical point theories, and in addition, a very good treatment of Morse theory. This topic is then related to the Morse theory of submanifolds in Euclidean space with a proof of the Morse Index Theorem and the Morse Inequalities. Witten's ideas on using eigenvalue theory of appropriate differential operators to prove the Morse Inequalities, however, is not mentioned, but the book does cover some relatively new topics not known in the sixties. For example, a nice proof of Rabinowitz's Mountain Pass Lemma is included. The second chapter of part two pertains to more advanced topics of Critical Point Theory. In the author's words, "Our original Minimax Principle located critical levels. Now we will look for more refined results that locate critical points." The authors describe the idea of Linking type and Bott Samelson type for critical points. The final chapter of part two deals with the Calculus of Variations treated in a rather formal way, although the chapter does end with a lively discussion of nonlinear eigenvalue problems using Sobolev spaces and ending with the Yamabe problem of global differential geometry relating to the deformation of scalar curvature to a constant. This last discussion is carried out a little tongue-in-cheek since the topic is really just the beginning of a rather elaborate theory in global differential geometry and the calculus of variations. For example, the authors never do say that they have solved the Yamabe problem in any single case, nor do they mention in any coherent way the related work of T. Aubin and R. Schoen. The whole idea of using Sobolev space, the embedding theorems, and associated regularity theory is pulled out of the hat without any motivation of coherent references to the literature. Obviously this topic is worth another new book. The authors could have added some excitement to the book by relating parts one and two to the recent work of H. Wente, who used equation 3.1.16 of the text derived in Chapter Three to discuss new surfaces of constant mean curvature. This nonlinear partial differential equation with associated boundary values could be studied fairly easily by the methods discussed in part two.

All in all, this voluome makes a very fine addition to the literature of both Differential Geometry and Critical Point Theory. In a recent conference celebrating the birthday of Stephen Smale, I chatted with the authors concerning possible extensions of the Calculus of Variations in the large to other problems in Global Geometry when the famous condition C of Palais-Smale was violated. Both authors had interesting views on this subject, and it is hoped that this idea, together with this Lecture Notes volume, will be extended to a more comprehensive book covering the same topics.

In the book by Mawhin and Willem, *Critical point theory and Hamiltonian systems*, the Calculus of Variations is used to study a specific physical problem from classical mathematical physics, namely, the periodic motions of finite-dimensional nonlinear Hamiltonian systems. The basic modern idea about studying such periodic solutions is to extend the Fourier theory approach to periodic motions of Hamiltonian systems using Sobolev's ideas on weak derivatives, thus obtaining a Hilbert space viewpoint for analyzing the infinite-dimensional geometry of nonquadratic functionals defined on these function spaces. This book develops the Calculus of Variations ideas that are needed for this study. The book consists of ten chapters equally divided between Calculus of Variations theory and the specific example of periodic motions. The book begins by studying minimization ideas using convexity and lower semicontinuity to obtain periodic solutions as an absolute minimum for nonautonomous second-order Hamiltonian systems. The book then moves on to more subtle ideas of periodic solutions as saddle points of appropriate functionals. Thus, in the latter chapters of the book, contemporary versions of Ljusterick-Schnirelman theory and Morse theory are used in connection with multiple periodic orbits under various physical conditions, such as prescribed energy or fixed period.

Various new ideas, for example, the Mountain Pass Lemma and the Morse-Ekeland Index, are also described. The book treats the special case of second-order Hamiltonian systems in special chapters throughout the book; a good idea since the geometry of the associated functionals is more transparent to analyze in these cases. Thus, in Chapter 1, solutions can be obtained as strict minimizers of an appropriate energy functional, and in later chapters, the nonlinear eigenvalue problems associated with periodic solutions are more easily described.

One difficulty with the book is the strange and interesting ambiguity of Hamiltonian systems reacting with critical point theory. Thus, physical ideas of integrable Hamiltonian systems, conservation laws, quasi-periodic motions, and the K.A.M. theorem are hardly mentioned even though these concepts are very fundamental for the current study of nonlinear Hamiltonian systems; while for Critical Point Theory the ideas described seem relatively unmotivated instead of coming from a coherent geometric picture of mechanics. Thus, the book focuses on finite-dimensional Hamiltonian systems and does not discuss infinite-dimensional systems at all, even though such an extension is very important in contemporary research. The book focuses on Hamiltonian systems connected with the theory of oscillations in engineering, but the ideas connected with periodic motions in celestial mechanics are hardly mentioned, even though they both are very basic and very classic. One exception occurs in Chapter 9.2, on p. 207, when periodic solutions of autonomous second-order systems near equilibrium are considered. But, the associated classical result associated with the Russian mathematician, Liapunov, is not described. Nor is the historical context of why this result is so important. The idea of periodic motions associated with Kepler's two body problem are not described, nor the context for the historic search for periodic motions in the more general three-bodies case.

All in all, with the above caveats, the book of Mawhin and Willem is a very fine introduction to the ideas of utilizing contemporary functional analysis for the classical physical problem of periodic motions of finite-dimensional Hamiltonian systems. Many other books on this topic are being written, and it is hoped that the volumes will not only popularize the subject but lead to a great flowering of this discipline utilizing the most contemporary results in mathematical analysis and topoplogy.

M. S. Berger
University of Massachusetts, Amherst

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Capacity theory on algebraic curves, by Robert S. Rumely, Lecture Notes in Mathematics, vol. 1378, Springer-Verlag, Berlin, Heidelberg, and New York, 1989, 437 pp., $37.50. ISBN 3-540-51410-4

If you asked a number theorist, an analyst, a logician, and an arithmetic geometer to define capacity theory, you would be likely to hear four different answers. To a number theorist, capacity theory has to do with the possible locations of sets of Galois conjugate algebraic numbers, as well as with criteria for power series to be rational functions. An analyst might discuss potential theory and extremal subharmonic functions. Logicians have found capacity theory to