machine-scheduling problems reduce to finding eigenvectors and eigenvalues of a matrix over a semiring; certain path-finding problems reduce to the solution of linear equations over an ordered structure.

Many combinatorial optimization problems assume, under such reformulation, the appearance of problems of linear algebra over an ordered system of scalars. Hence we may look to the highly-developed classical theory of linear algebra over the real field to give us hints as to how we might approach these problems, or, if appropriate adaptations of classical techniques cannot be found, we have a well-defined research program to elucidate the theory of linear algebra over such ordered structures, and to see how far the algorithms and duality principles, familiar to us from linear and combinatorial optimization over the real field, extend to more general structures.

These questions have stimulated a good deal of research over the last twenty-five years. From a few isolated publications by one or two researchers in the late 1950s, the subject has matured into an identifiable branch of applicable mathematics with an international following.

The author has made a comprehensive survey of this work, to which he himself has notably contributed. His book is divided into two sections. In the first, a systematic theory of ordered algebraic structures is presented; in the second, the subject of linear algebraic optimization is explored. The topics discussed are generally, though not exclusively, of one or two kinds: either they relate to the properties of matrices over such of the algebraic structures as are rich enough to permit matrix multiplication, or they analyse the extent to which analogues of familiar linear and combinatorial optimization problems may be formulated and algorithmically solved for general ordered algebraic structures.

Because it so comprehensively reviews a literature which is widely scattered throughout a great variety of journal articles, this book will be a valuable addition to the library of any researcher seriously interested in this field.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

Pseudo-differential operators, by Hitoshi Kumano-go, The MIT Press, Cambridge, Massachusetts, 1982, xviii + 455 pp., $60.00. ISBN 0-2621-1080-6.

Introduction to the Fourier transform and pseudo-differential operators, by Bent E. Petersen, Pitman Advanced Publishing Program, Boston, Massachusetts, 1983, xi + 356 pp., $48.00. ISBN 0-2730-8600-6.

Functions of a finite set of selfadjoint operators, commuting with each other, can be defined through spectral theory. Pseudo-differential operators come into action when the need to represent functions of noncommuting operators arises. More specifically, let us consider the Hilbert space $L^2(\mathbf{R}^n)$, and, for every $j = 1, 2, \ldots, n$, the operators q_j and p_j defined by $(q_j u)(x) = x_j u(x)$ and $(p_j u)(x) = (1/i)(\partial u/\partial x_j)$: these are selfadjoint operators on $L^2(\mathbf{R}^n)$, the domain of which contains Schwartz' space $\mathscr{S}(\mathbf{R}^n)$ of smooth functions rapidly going to zero at infinity. That they do not commute is expressed by Heisenberg's relations $[q_j, q_k] = [p_j, p_k] = 0$; $[q_j, p_k] = i\delta_{jk}$. Defining functions of a commuting set of operators chosen among the linear combinations, with real coefficients, of the operators $q_1, \ldots, q_n, p_1, \ldots, p_n$ leaves no room for choice. In particular, if $a \in \mathbf{R}^n$ and $b \in \mathbf{R}^n$,

$$e^{i(\langle a,q\rangle + \langle b,p\rangle)} = \exp i\Sigma(a_j q_j + b_j p_j)$$

is uniquely defined, say by Stone's theorem on one-parameter groups of unitary operators, as

$$(1) \qquad (e^{i(\langle a,q\rangle + \langle b,p\rangle)}u)(x) = e^{i\langle a, x+b/2\rangle}u(x+b).$$

Now the Fourier transformation makes it possible to decompose any reasonable function f on $\mathbf{R}^n \times \mathbf{R}^n$ as a superposition of functions $(x,\xi) \mapsto e^{i(\langle a,x\rangle + \langle b,\xi\rangle)}$; thus any function f shall be assigned a (generally unbounded) operator $f(q,p)$ as soon as one has set a rule for letting operators correspond to special functions of the above-mentioned type and decided to extend it in a linear way. The Weyl assignment is such that the function $e^{i(\langle a,x\rangle + \langle b,\xi\rangle)}$ gives rise to the operator $e^{i(\langle a,q\rangle + \langle b,p\rangle)}$; in the "standard" assignment, it gives rise to the product operator $e^{i\langle a,q\rangle}e^{i\langle b,p\rangle}$. Of course, the notation $f(q,p)$ is meant to convey the idea (at best meaningful in an approximate sense) that one has substituted the operator q_j (resp. p_j) for the real variable x_j (resp. ξ_j) in the function f: the operator $f(q,p)$ is called the pseudo-differential operator with symbol f, though mathematicians sometimes prefer to denote it as $f(X,D)$ when the standard rule is used, and $f^W(X,D)$ under the Weyl assignment. The defining formulas are as follows:

$$(2) \qquad (f(X,D)u)(x) = (2\pi)^{-n}\iint f(x,\xi)e^{i\langle x-y,\xi\rangle}u(y)\,dy\,d\xi$$

and

$$(3) \qquad (f^W(X,D)u)(x) = (2\pi)^{-n}\iint f\left(\frac{x+y}{2},\xi\right)e^{i\langle x-y,\xi\rangle}u(y)\,dy\,d\xi.$$

Formula (2) is more consistent with the usual way people write a differential operator P as

$$P = \sum a_\alpha(x)\left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha$$

(derivatives first), since it gives to such an operator the symbol $p(x,\xi) = \Sigma a_\alpha(x)\xi^\alpha$: this remark may also explain the terminology "pseudo-differential operator" (you may call them ψ.d.o's when you have gotten acquainted with them). As a matter of fact, there is nothing in formulas (2) or (3) that would impose any limitation on the operators one gets, as any linear operator, say from $\mathscr{S}(\mathbf{R}^n)$ to its dual, can be represented as $f(X,D)$ or $g^W(X,D)$ for suitable f or $g \in \mathscr{S}'(\mathbf{R}^{2n})$; the subject of ψ.d.o's really begins when only symbols which are nice and not too large are considered. The standard rule (2) has been traditional with people in partial differential equations for years (though the trend is now reversing for some very good reasons) and is the choice made in the two books under review. The symbols used therein lie in Hörmander's classes $S^m_{\rho,\delta}$, where $f \in S^m_{\rho,\delta}$ means that f is smooth and satisfies estimates

$$(4) \qquad |D_x^\alpha D_\xi^\beta f(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\beta|+\delta|\alpha|};$$

here ρ and δ are constants such that $0 \leqslant \delta \leqslant \rho \leqslant 1$, $\delta < 1$, and m is allowed to depend on f: only the case when $\delta = 0$ and $\rho = 1$ is really natural, but the flexibility allowed by the more general case is helpful in some situations. Actually, important and difficult problems in partial differential equations have been solved just by the introduction of a well-adapted class of symbols for which a symbolic calculus was made available (e.g. R. Beals' and C. Fefferman's extension of the Nirenberg-Treves results on the local solvability problem in the principal type case). Symbols whose order m is highly negative, and associated operators, are considered as negligible in ψ.d.o. theory; for instance, formulas (2) and (3) are related by the fact that $f^W(X,D) = g(X,D)$ provided that

$$g(x,\xi) = \left(\exp \frac{1}{2i} \sum \frac{\partial^2}{\partial x_j \partial \xi_j}\right) f(x,\xi)$$

(Stone's interpretation of the exponential again); this may be converted into the asymptotic expansion

$$(5) \qquad g(x,\xi) \sim \sum \frac{1}{\alpha!}\left(\frac{1}{2i}\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\alpha f(x,\xi),$$

where, in view of (4), the terms on the right-hand side become more and more negligible in the case when $\delta < \rho$. Thus, working with the $S^m_{\rho,\delta}$ classes, one gets the same operators whether one uses (2) or (3); (5) is typical of various asymptotic expansions, which together constitute the so-called *symbolic calculus*. The most important such formula is the one that expresses the symbol $f \circ g$ of the composition of the operators $f(X,D)$ and $g(X,D)$; it reads

$$(6) \qquad (f \circ g)(x,\xi) \sim \sum \frac{1}{\alpha!}\left(\frac{\partial}{\partial \xi}\right)^\alpha f(x,\xi) \cdot \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^\alpha g(x,\xi);$$

again (4) justifies the view that the successive terms may be considered as smaller and smaller if $\rho > \delta$. A key notion, introduced by Hörmander and very much related to the development of ψ.d.o. theory, is that of wavefront set: given a distribution u on $\mathbf{R}^n$, it is possible to describe the set where u is not smooth not as a (closed) subset of the x-space $\mathbf{R}^n$, but as a subset of $\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ conical in the sense that it is invariant under the dilations $(x,\xi) \mapsto (x,\lambda\xi)$ with $\lambda > 0$. This represents in a way a reaction against Heisenberg's uncertainty principle (which asserts essentially that there can be no definition of the support of u as a subset of the "phase space" $\mathbf{R}^n \times \mathbf{R}^n$), and this notion has proved crucial in the modern development of partial differential equations: the reason for this is that it allows one, to a certain extent, to describe things as happening in the phase space rather than downstairs in $\mathbf{R}^n$, a fundamental breakthrough as it is certainly in the phase space that the interesting geometry takes place.

The matters discussed so far are the core to any elementary introduction to ψ.d.o. theory. They constitute much of the book by Bent E. Petersen, which has been organized at the same time as a textbook on distribution theory. The applications include a short discussion of the Dirichlet problem, partly as a motivation for studying the sharp Gårding inequality, and Hörmander's theorem on the propagation of singularities for ψ.d.o's of principal type with a real principal symbol. Its usefulness as a textbook is enhanced by the presence of numerous easy exercises and a few scattered references to results in partial differential equations not developed in the text. My only reservation would be about the choice of certain topics (a great deal of soft analysis in the chapter on local existence, while I would have found other matters more urgent), but this is, after all, a question of taste.

Pseudo-differential operators did *not* make their appearance in partial differential equations as a position-and-momentum functional calculus. Their ancestors must be traced through the development of classical potential theory, so that Newton, Laplace, Poisson and Green may have been the true founders of the field. In potential theory, operators occur whose kernels have bad singularities on the diagonal; these singularities, however, are of a quite well-defined nature. Calderón and Zygmund initiatied, in the fifties, the modern era of ψ.d.o's: not only did they introduce the capital notion of symbol (for classes of operators initially characterized by their kernels so as to resemble those of potential theory), but, moreover, Calderón was the first to show the applicability of ψ.d.o's in problems about *general* differential operators. Ever since that time, ψ.d.o's have been helpful in all sorts of problems in partial differential equations: local solvability and hypoellipticity, Cauchy and boundary-value problems, the spectral theory of differential problems, complex analysis [**1**]; the related field of hyperfunctions and analytic microlocal analysis, initiated by Sato, has grown and matured. The bibliographical notes in H. Kumano-go's book constitute a very helpful guide to the (quite formidable) literature up to 1980. More recent developments include J. M. Bony's calculus of paradifferential operators and applications to nonlinear partial differential equations.

On the other hand, some essential ideas in ψ.d.o. theory can be grasped best through its connection to other fields, in particular, quantum mechanics and noncommutative harmonic analysis. The discussion that follows is intended to show, at the same time, that ψ.d.o's may also have a more and more important role to play outside the field of partial differential equations. The "grand scheme" below has been phrased in many different versions and should be ascribed to the founding fathers of quantum mechanics. Let M be a smooth manifold, H a complex Hilbert space, Γ a connected Lie group of C^∞ transformations of M, say acting transitively on M. Let L be a real vector space consisting of selfadjoint operators on H, and for every $A \in L$ let $\sigma(A)$ be a smooth real-valued function on M; call $\sigma(A)$ the symbol of A, and assume that σ is linear and one-to-one. Finally, for every $\Psi \in \Gamma$ let U_Ψ be a unitary transformation of H: assume that U is a projective representation of Γ (i.e., $U_\Psi U_{\Psi'} = \lambda U_{\Psi\Psi'}$ with $\lambda \in \mathbf{C}$, $|\lambda| = 1$) and the symbol map is covariant under that representation: this means that if $A \in L$, so does $U_\Psi A U_\Psi^{-1}$, and $\sigma(U_\Psi A U_\Psi^{-1}) = \sigma(A) \circ \Psi^{-1}$; to close the list of demands, assume that if (Ψ_t) is a one-parameter subgroup of Γ, then, for some choice of the unit complex number λ_t, one has $\lambda_t U_{\Psi_t} = \exp itA$, with $A \in L$, so that one can define $f = \sigma(A)$. Now, denoting by X_f the vector field that is the generator of (Ψ_t) and using the covariance property to compute the symbol of $(d/dt)(e^{itA}Be^{-itA})$ $(t = 0)$ in two different ways, one gets, if $\sigma(A) = f$ and $\sigma(B) = g$, the relation $\sigma(i[A,B]) = -X_f g$.

The grand scheme belongs to more than one trade, depending on the fixtures and portable things among its ingredients. One may start with the manifold M that is the phase space of a certain classical system; as such, it has a symplectic structure, which allows one to define Poisson brackets. The founders of quantum mechanics, H. Weyl among them, had to face the quantization problem, which just means completing the picture in such a way that $-X_f g = \{f,g\}$: then, given the hamiltonian of the system (this is just a function f on M), the Hamilton-Jacobi equations that describe its classical evolution in time could be replaced by a Schrödinger equation. Of necessity, the transformations that belong to Γ must preserve the symplectic form on M; such transformations are called canonical. It is well known (Van Hove's theorem) that for no fixed quantization rule σ (or rather σ^{-1}) can all canonical transformations belong to Γ. On the other hand, it seems to be less known that quite miscellaneous groups Γ do fit in this scheme provided σ is subject to choice. Consider the case when $M = \mathbf{R}^n \times \mathbf{R}^n$ with the symplectic form $\sum dx_j \wedge d\xi_j$: then M acts on itself by translations, and (1) defines a projective representation of M in $L^2(\mathbf{R}^n)$, the Heisenberg representation; one does get covariance with either of the quantization rules (2) and (3) (and many others too). With the Weyl rule (3), one gets as a bonus covariance under the action on M of $\mathrm{Sp}(n,\mathbf{R})$, the group of linear canonical transformations of M: the associated representation is the celebrated metaplectic representation. The reviewer has recently suggested a quantization rule meaningful in cases when M is a riemannian symmetric space and Γ is its isometry group; besides the possible applications of such calculi in partial differential equations, one gets a ψ.d.o. interpretation of the Radon transformation as a connecting link between various calculi.

In the theory of group representations, the grand scheme plays a role, explicit or not, in Kirillov's method of orbits and Kostant's and others' quantization and polarization theory. There, only the group Γ is given to start with, not M: the connection of all this with ψ.d.o's has been noted by R. Howe. Another classical exercise in harmonic analysis which is obviously very much related to ψ.d.o's is the problem of decomposing the tensor product of a representation by the adjoint representation.

The next (and last) interpretation of the quantization scheme will again take us to matters that have been more central to the development of ψ.d.o's and their use in partial differential equations. Fix $M = \mathbf{R}^n \times \mathbf{R}^n$ and the standard quantization rule (2) (Weyl's rule would work in the same way). Now, as shown by (6), in general, one does not have $\sigma(i[A,B]) = \{\sigma(A),\sigma(B)\}$ in general. Recalling that

$$\{f,g\} = \sum\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right),$$

one can see, on the other hand, that $\{\sigma(A),\sigma(B)\}$ is just what one would get by retaining in the right-hand side of (6) only the terms with $|\alpha| \leqslant 1$. Choosing classes of symbols characterized by inequalities more or less in the shape of (4) (with $\rho > \delta$) gives substance to this asymptotic point of view, though of course $\{f,g\}$ can be considered as the main term in $i(f \circ g - g \circ f)$ only under some nondegeneracy (principal type) assumptions. Physicists have another, more formal, point of view, as they would quantize f as $f(X, \hbar D)$ rather than $f(X,D)$, which makes powers of Planck's constant apparent in (6). Now that room has been made for error terms, the discussion of the grand scheme makes it plausible that one can define U_Ψ (i.e. *quantize* Ψ), in an approximate sense, for rather general canonical transformations Ψ. This was, indeed, done by Egorov in 1969 for canonical transformations that are homogeneous of degree one in the ξ-variable; one gets operators U_Ψ which are, if not unitary, at least nearly invertible, and complete asymptotic expansions make it possible to get error terms as small as one pleases. An elementary case occurs when Ψ is the map induced on $\mathbf{R}^n \times \mathbf{R}^n$, interpreted as the cotangent bundle of $\mathbf{R}^n$, by a diffeomorphism ψ of the base space. Connections between the symbolic calculus of operators and quantum mechanics seem to have been brought forward in a large part by the Russian school, especially Gelfand, Berezin, and Maslov; Leray did much, with his Lagrangian Analysis, to publicize and clarify this trend of ideas. In 1968–1970, Hörmander introduced and developed his calculus of Fourier integral operators; besides a symbol, the definition of such an operator requires the introduction of a "phase function" built from a canonical transformation. One of their most important properties is the following: just as, for a given ψ.d.o. A, and every distribution u to which it can be applied, the wave front of Au is included in that of u (operators that satisfy this property are called pseudo-local), Fourier integral operators move the wave front set according to the canonical transformation built in their phase function. This is especially important when the need arises to construct operators which, by nature, propagate singularities, in particular those that describe the solution of an evolutionary problem of hyperbolic type. Indeed, the first example of a Fourier integral operator had been introduced by Lax in precisely such a problem.

Besides being much more than just an introduction to the subject of ψ.d.o's and Fourier integral operators, the book by Kumano-go is an exposition of what his views were on the subject; as such, it is now invaluable. There is considerable material in the book, much of it treated in an original way. To give an idea of the broad range of applications considered, let us mention: elliptic complexes and the Atiyah-Bott-Lefschetz theorem; Hörmander's theorem on the hypoellipticity of "sums-of-squares" second-order operators, in Kohn's way of proof; general elliptic boundary-value-problems (Shapiro-Lopatinski conditions); Calderón's uniqueness in the Cauchy problem; the construction of fundamental solutions, with an original method, for initial-value problems of hyperbolic type. The material covered reflects in a way Kumano-go's work; on the other hand, as I said earlier, the book contains a very useful guide to the literature, written in a systematic way, from which the reader could get a fair idea of what directions partial differential equations have taken in recent years. Care has been taken to adhere to fixed notations and to display a complete list of them as well as an index. However, even though preliminary material like tempered distributions and the Fourier transformation is covered (at a brisk pace) in the first chapter, I am afraid that newcomers to the field may find that the book, in general, makes hard reading. One of the drawbacks of the author's highly respectable constant emphasis on hard analysis is that essential ideas are sometimes concealed so as to appear as just technicalities. On the other hand, there *are* great ideas in the book. The one that makes it most original is the recurrent use of multiple symbols, a notion introduced by Kumano-go (though Friedrichs had introduced very early not-so-multiple symbols). The idea is to replace the phase space $\mathbf{R}^n \times \mathbf{R}^n$ by $(\mathbf{R}^n \times \mathbf{R}^n)^\nu$ for large ν. Then, the quantization rule is chosen so that

$$\exp i \sum_{k=1}^{\nu} \left(\left\langle a_k, x^k \right\rangle + \left\langle b_k, \xi^k \right\rangle \right)$$

give rise to the operator $e^{i\langle a_1,q\rangle} e^{i\langle b_1,p\rangle} e^{i\langle a_2,q\rangle} \ldots e^{i\langle b_\nu,p\rangle}$; in particular, if f_k is the standard symbol of A_k, a multiple symbol of the product $A_1 \ldots A_\nu$ is the function

$$f(x^1,\xi^1,x^2,\ldots,\xi^\nu) = \prod_{k=1}^{\nu} f_k(x^k,\xi^k).$$

Any multiple symbol f can be contracted to an ordinary symbol g that would yield the same operator; careful estimates of g, depending on ν in as uniform a way as is possible, are given. The idea of multiple symbol is very reminiscent of the use of Trotter's formula

$$\exp -t(-\Delta + V) = \lim_{\nu \to \infty} \left(e^{(t/\nu)\Delta} e^{-(t/\nu)V} \right)^\nu$$

to derive the Feynman-Kac formula from the expression of $(e^{(t/\nu)\Delta} e^{-(t/\nu)V})^\nu$ as the product of a great number of convolution and multiplication operators. Together with a similar construction relative to Fourier integral operators, it plays a crucial role in the author's study of hyperbolic systems.

Kumano-go's book should not be compared to Treves' two volumes at Plenum, which appeared at the same time, and whose geometrical insight and enjoyable style could hardly be surpassed. Still, it is a book with a great wealth of material; also, it will be found helpful by people interested in knowing better Kumano-go's own contributions and insights in the field of pseudo-differential operators.

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