BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 3, May 1983

0273-0979/83/0000-1401/$01.75

Group analysis of differential equations, by L. V. Ovsiannikov, Academic Press, New York, 1982, xvi + 416 pp., $54.00. ISBN 0-1253-1801-1

This book is an attempt by an applied mathematician to come to terms with Sophus Lie's legacy: The use of group theory and differential geometry to study differential equations. Lie's work was almost forgotten after 1914: It has only begun to be revived in recent years, and there is much in the turn-of-the century literature which has still to be incorporated into contemporary mathematics. It is also remarkable that this renaissance in the topic which constitutes the very historical foundation of Lie group theory has come about largely because a few applied mathematicians, physicists, chemists and engineers have realized that there is a common group-theoretic and geometric structure to what they do, and that Lie and some of his immediate successors (above all, E. Vessiot and E. Cartan) developed, in terms of the mathematics of their day, a program which has major implications for their own science. Before I comment on the book itself, I want to present some remarks about the mathematical context of Lie's work on differential equations and group theory.

His starting point was Galois' work on the relation between group theory and the solution of polynomial equations in one variable. Indeed, Lie saw himself, in a romantic 19th century style, as the successor to Abel and Galois who would extend their ideas to differential equations. Let us then begin with Galois theory. Consider a polynomial

$$P(x) = a_n x^n + \cdots + a_0$$

in one complex variable x, whose coefficients lie in a subfield K of the complex numbers C. Let $S(P)$ be the set of $x \in C$ such that $P(x) = 0$. Let $K(P)$ be the subfield of C generated by K and $S(P)$, and let $G(K, P)$ be the group of automorphisms of $K(P)$ which leave each element of K fixed. Nowadays, "Galois theory" is considered to be the study of the relation between the properties of $K(P)$ and $G(K, P)$. The classical questions of "solvability by radicals" of the equation $P(x) = 0$ is related to the solvability, in the group-theoretic sense, of $G(K, P)$. However, there is another point of view which is fundamental for understanding Lie's approach. Consider $G(K, P)$ as a transformation group on $K(P)$. It leaves $S(P)$ invariant, hence, induces a transformation group action on $S(P)$. How does one characterize this transformation group intrinsically? It can be done in the following way:

> For each integer $r > 0$, let $S(P)^r$ and C^r be the Cartesian product of r copies of $S(P)$ and C. $G(K, P)$ extends as a transformation group on these Cartesian product spaces. There is a set I of polynomial maps $C^r \to C$ (the integer r and the construction of I depending on K and P) such that $G(K, P)$, as a transformation group on $S(P)$, is identified with the set

of all invertible maps $g\colon S(P) \to S(P)$ which leave invariant these polynomials.

Lie's fundamental idea was to consider a differential equation DE replacing P; the set of all solutions $S(\mathrm{DE})$ of the differential equation replacing $S(P)$; transformation groups of mappings on $S(\mathrm{DE})$ replacing $G(K, P)$; I was replaced by what he called *differential invariants* attached to DE. However, it is not clear in my mind whether he meant these differential invariants to be canonically attached and/or calculated in terms of DE, or prescribed separately. Perhaps he meant different interpretations in different contexts!

Some of this material can now be stated most clearly in the language of Ehresmann's jet calculus. (Lie himself used a concept, which in my commentary [**1**] on his work I called "mapping element spaces", but which is geometrically identical to that of the jet spaces.) Let X and Y be finite-dimensional, paracompact, C^∞ manifolds, and let $J^m(X, Y)$ be the space of m-jets of mappings $X \to Y$. DE, if given in the classical style as an mth order differential equation to be solved for maps with domains in X and range in Y, can be considered as a submanifold of $J^m(X, Y)$. A mapping $\alpha\colon U \to Y$ between an open subset U of X and Y is a solution of DE if

$$j^n(\alpha)(U) \subset \mathrm{DE}.$$

Thus, $S(\mathrm{DE})$ can be considered as a collection of sufficiently smooth mappings between open subsets of X to Y. The jet prolongation process enables one to realize these maps also as maps from open subsets of X to $J^n(X, Y)$, $n = 0, 1, \ldots, \infty$.

The Galois group of the DE is now to be considered as a certain group of transformations acting on $S(\mathrm{DE})$, and preserving certain "differential invariant" relations. The theory is simplest if one restricts attention to transformations on $S(\mathrm{DE})$ arising from pseudogroups, in the sense of Ehresmann and Spencer [**2**], which act on the jet spaces. However, making all this precise and workable inevitably involves complicated machinery and formalism, and has not yet been done in any setting which is sufficiently complete and accessible to encompass most of the possible applications. What one must do at the present time is to pick out a relatively small domain of application and develop Lie's ideas in a form especially tailored for the problem at hand. What would be very desirable, and useful for many purposes in both pure and applied mathematics, would be a codification of this material in terms of standard mathematics in a form general enough to be used in a wide variety of situations.

Of course the work of Lie, Vessiot and Cartan took place before the development of abstract group theory. (From the modern point of view, their work is a theory of transformation groups, or their "sheaf-theoretic" version, the pseudogroups.) There have since then been many attempts at a codification in terms of such an abstract group theory. What has of course been very successful is the development of the theory of what are now called "Lie Groups", and their associated Lie Algebras and geometric actions as transformations on finite-dimensional manifolds. These are groups modelled on the finite-dimensional differentiable manifolds; the success of this theory is basically due to the perfection of the necessary tools of analysis and algebra, especially the availablity with sufficiently useful hypotheses of the Implicit Function Theorem and the Existence-Uniqueness of Ordinary Differential Equations. Now these two theorems extend to Banach spaces if the hypotheses are chosen correctly: Thus there is a possibility (perhaps first recognized clearly by Garrett Birkhoff [**3**]) of developing a useful theory of groups modelled on Banach spaces. However, it is known that many groups which are very natural from the geometric/Lie/Cartan/Ehresmann/Spencer point of view cannot be given such a Banach structure. They can often be modelled on Fréchet spaces—the next step in the functional analysis hierarchy beyond Banach spaces—but the two basic existence theorems break down for such spaces.

This book, a translation from Russian of one first published in 1978, has already been a seminal influence on the development and propagation of the Lie-theoretic tools in the applied mathematics community. It has three components:

(A) A development, with partial proofs, of a theory of abstract groups and transformation groups modelled on Banach spaces, following the lead of G. Birkhoff.

(B) A restatement in reasonably accessible form of much material from the classical papers of Lie, Bäcklund, Tresse and Vessiot.

(C) A program, partially carried out for certain differential equations of interest in physics and applied mathematics, of calculation of at least part of the differential equations of the Galois group mentioned above.

However, the reader who has some familiarity with developments in contemporary geometry—and this knowledge is now becoming widespread among the more mathematically oriented researchers in the physical sciences—will be disappointed that the manifold-theoretic ideas are expressed and used in such a rudimentary form. A formalism which is a primitive version of Ehresmann's theory of jets and prolongations of mappings is developed and used. Differential geometers will notice that the powerful techniques of Cartan—the theory of Exterior Differential Systems, Infinite Lie Groups, and the Equivalence Problem, which Cartan developed in the period 1895–1915 precisely to carry out Lie's program—plays no role. The author acknowledges this gap in the Preface, and suggests that his book can serve as an introductory account which can stimulate further work. In this he has succeeded: The ball is now in the court of the community of differential geometers to finally codify our magnificent heritage in the geometric-Lie theory of differential equations and its applied ramifications, which the author has brought to our attention in this useful book.

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