Integral formulas are difficult to discuss in just one chapter. For one thing, there are many different formulas; for another, their construction and use has remained a rather technical enterprise. Krantz limits himself to constructing and estimating the "Leray-Stokes" formula of Henkin in the case $n = 2$, a choice which strikes a good compromise between generality and technicality.

As a text, this book should be excellent for a second course on complex analysis. It covers many of the basic results and connects them up with harmonic analysis and P.D.E.; and the final three chapters provide an introduction to more specialized material.

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Hardy spaces on homogeneous groups, by G. B. Folland and E. M. Stein, Mathematical Notes, Princeton Univ. Press, Princeton, N. J., 1982, 284 pp., $13.00. ISBN 0-6910-8310-X

"The original and basic concept of functional analysis is that of an operator. Just as an analytic function has its complete domain of definition, so an operator has a complete set of spaces on which it can be examined." This maxim is from the 1966 survey article of S. G. Krein and Yu. I. Petunin on interpolation spaces [**5**], who also refer to "the victory of the operators over the spaces." For a nice example of this phenomenon, consider some recent results of Marshall, Strauss, and Wainger [**6**] concerning the nonlinear Klein-Gordon equation

$$\text{(NLKG)} \qquad v_{tt} - \Delta v + v + \lambda |v|^{\alpha} v = 0$$

and its linearization at $v = 0$,

$$\text{(KG)} \qquad u_{tt} - \Delta u + u = 0.$$

Here Δ is the Laplace operator on $\mathbf{R}^n$, and one assumes $\alpha > 0$, $\lambda > 0$. Given a function f in $L^2(\mathbf{R}^n)$, there is a unique (weak) solution $u(x, t)$ to (KG) with initial data $u(x, 0) = 0$ and $u_t(x, 0) = f(x)$. Let $T_t\colon f \to u(\cdot, t)$ be the operator which takes initial velocity f into position $u(\cdot, t)$ at time t. The problem is to construct a finite energy solution to (NLKG) which is asymptotic in the energy norm as $t \to -\infty$ to $T_t f$. As was shown previously by Strauss, this can be reduced to the problem of obtaining certain bounds (in terms of t) for the norm of the linear operator T_t from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, with $1/p + 1/q = 1$. This is carried out in [**6**] by using results of Fefferman and Stein about the operator $(1 - \Delta)^{is}$, $s \in \mathbf{R}$, on the space BMO, together with the Stein interpolation theorem applied to a holomorphic family of operators T_t^{α} $(\alpha \in \mathbf{C})$ containing T_t. The result is that the nonlinear scattering problem at $t = -\infty$ has a solution when the exponent α in (NLKG) and the space dimensionality n satisfy $4/n \leqslant \alpha \leqslant [4/(n - 1)]$. In particular, the physically interesting case $n = 3$, $\alpha = 2$ is included in this treatment.

This example illustrates the fruitfulness of using such "exotic" spaces as BMO, together with interpolation theory, to prove results for the more familiar L^p spaces. An additional ingredient in the example is that the operators considered are all functions of the selfadjoint, translation-invariant operator Δ. Now in general, if $\mathfrak{L}$ is a selfadjoint operator on an L^2 space, with spectral resolution

$$\mathfrak{L} = \int_{-\infty}^{\infty} \lambda \, dE_\lambda,$$

then for any bounded Borel function φ on $\mathbf{R}$ the operator

$$\varphi(\mathfrak{L}) = \int_{-\infty}^{\infty} \varphi(\lambda) \, dE_\lambda$$

is bounded on L^2, with norm $\|\varphi\|_\infty$ (the essential supremum of $|\varphi|$ relative to the maximal spectral measure class of $\mathfrak{L}$). By the general principle enunciated above, this definition of $\varphi(\mathfrak{L})$ via the spectral theorem is analogous to the definition of an analytic function such as the Euler gamma function by an integral formula with a limited domain of convergence. A complete description of the properties of $\varphi(\mathfrak{L})$ requires determining, e.g., the values of (p, q) so that $\varphi(\mathfrak{L})$ is bounded from L^p to L^q, and calculating the operator norm. The standard way to do this is to prove "endpoint" estimates and then to invoke the theory of interpolation spaces [**1**]. The "endpoint" estimates in turn are typically obtained using another realization of the operator $\varphi(\mathfrak{L})$, e.g., as an integral operator whose kernel has singularities of some specific type.

A class of operators $\mathfrak{L}$ for which the above program is particularly natural are the generators of *symmetric diffusion semigroups* [7]: $\mathfrak{L} \geqslant 0$, and the operators $e^{-t\mathfrak{L}}$ for $t > 0$ preserve pointwise positivity and are contractions on L^p, $1 \leqslant p \leqslant \infty$. The classic example of such an operator $\mathfrak{L}$ is the Laplace-Beltrami operator on a compact Riemannian manifold. Here the property of hypoellipticity ("Weyl's lemma") gives the selfadjointness of $\mathfrak{L}$, and the maximum principle gives the pointwise positivity and L^p-contraction property. A significant generalization of this situation has been intensively studied in recent years, in which $\mathfrak{L}$ is a *degenerate elliptic* operator of the form $-\sum X_i^2$, where $X_1, \ldots, X_n$ are real vector fields on a manifold whose iterated commutators $[X_k, \ldots [X_2, X_1], \ldots]$ for $1 \leqslant k \leqslant l$ span the tangent space to the manifold at each point. These operators are hypoelliptic, by a theorem of Hörmander, and the corresponding "heat" operator $\partial_t + \mathfrak{L}$ satisfies a version of the maximum principle due to J. M. Bony. The prototypical case is that of a "stratified" l-step nilpotent Lie group G (the most regular of the class of "homogeneous groups" treated by Folland and Stein). Here $X_1, \ldots, X_n$ are left-invariant vector fields which generate the Lie algebra $\mathfrak{g}$ of G. The "homogeneity" condition requires that the map $X_i \to rX_i$, for fixed $r > 0$, extend to a Lie algebra automorphism δ_r of $\mathfrak{g}$. In this case $\det(\delta_r) = r^Q$, where the integer Q is called the "homogeneous dimension" of G. The operator $\mathfrak{L}$ is selfadjoint on $L^2(G)$, and the "heat semigroup" $e^{-t\mathfrak{L}}$, $t > 0$, is given by right convolution with a positive C^∞ function h_t on G (this function is a natural analogue of the classical Gauss kernel on $\mathbf{R}^n$). Thus the general machinery of the theory of diffusion semigroups is available, together with the theory of analysis and representations of nilpotent Lie groups, to attack the problem of finding "a complete set of spaces on which $\mathfrak{L}$ can be examined."

Having provided background and motivation for some of the topics treated by Folland and Stein, the reviewer feels that it is best now simply to refer the reader to the lucid introductory sections, notes, and references in their book for a guide to this latest chapter in the story of Hardy spaces. In fact, the theory of these spaces has been the subject of much outstanding expository writing, which treats in depth the major theorems and the technical machinery of the subject (cf. C. Fefferman [**3**] and Coifman and Weiss [**2**]). Further attempts along these lines in this review would be redundant. For those readers who are familiar with the situation on $\mathbf{R}^n$ but have some uneasiness about the intrusion of Lie groups into "classical analysis" (perhaps akin to the attitude of physicists in the 1930s who hoped to stamp out the "group pest" that was beginning to infect quantum mechanics), the reviewer is of course happy to recommend his own expository efforts in this area [**4**]. Such readers should perhaps be warned that some of the technical underpinnings of the theory erected by Folland and Stein include the recent convolution factorization theorem of Dixmier and Malliavin for $C_c^\infty(G)$, and the theorem of Helffer and Nourigat characterizing homogeneous hypoelliptic operators on a homogeneous group G in representation-theoretic terms.

Having abdicated the responsibility of defining the Hardy spaces $H^p(G)$ for the reader (where G is a stratified nilpotent Lie group), the reviewer would nonetheless like to give a sample of the results to be found in this highly detailed and carefully written book. The following multiplier theorem, due jointly to Stein and A. Hulanicki, fits most naturally within the framework of this review: Let Q be the homogeneous dimension of G, $\mathfrak{L}$ a subelliptic Laplacian on G as above. Suppose φ is a function of class C^s on $(0, \infty)$, r is a positive integer, and $s > r + (3Q/2) + 2$. Assume that $\sup_{\lambda>0} |\lambda^j \varphi^{(j)}(\lambda)| < \infty$ for $0 \leqslant j \leqslant s$. Then the operator $\varphi(\mathfrak{L})$ is bounded on $H^p(G)$ in the range $Q/(Q + r) < p < \infty$. In particular, the operator $(1 + \mathfrak{L})^{is}$ is bounded on $H^p(G)$ for $s \in \mathbf{R}$ and $0 < p < \infty$.

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