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Applications of number theory to numerical analysis, by Loo Keng Hua and Yuan Wang, Springer-Verlag, Berlin, Science Press, Beijing, 1981, 241 pp., $39.00.

What is number theory good for? No one doubts that many branches of mathematics owe their existence to—or, at least, were strongly stimulated by —problems of the "real world", like those of physics, engineering, etc. Familiar examples are the calculus and the theory of differential equations needed in celestial mechanics; partial differential equations that are indispensable in hydrodynamics and so on. But number theory? Often number theorists, when challenged by our first question (usually asked by nonmathematicians) feel obligated to convince the questioner that number theory also can be useful. Sometimes its applications in problems of crystallography and, more recently, in cryptography are mentioned. Why it should be necessary to point out a "usefulness" in the commonly understood sense for number theory is something of a mystery to this reviewer. It appears quite certain that Diophantus, or Fermat, or Gauss studied this field of human knowledge because of its intrinsic interest and its peculiar beauty—and they really did not care one way, or the other, whether their elegant theorems would, or would not have "useful" applications.

Be that as it may, it turns out that like so many other branches of mathematics, developed by the "purest" of mathematicians, also number

theory *does* have applications outside itself. In addition to cryptography, or to the numerous problems of physics concerned with lattices (crystallography is just one of them; the study of perfect gases is another one; see [2]), one may list the many applications of number theory to the theory of computers (see, e.g., [9, Vol. 2]), the generation of random numbers (see [15, or 4]), and many more.

Two relatively recent publications, [23] edited by Zaremba and the present book by Hua and Wang, called attention to yet another great field of number theoretic application, namely numerical analysis.

At present, the number theorist who feels compelled to justify to the world the love for his field by the latter's "usefulness" can proudly point to the need for sophisticated number theory in [23] and in Hua and Wang, but also in Knuth's *Art of computer programming* [9], numerous papers by Dieter (see, e.g., [23, pp. 287–317] and [5]), and many more.

While Zaremba's collection of papers [23] discusses applications of number theory to numerical analysis understood in a rather broad sense, the book by Hua and Wang concentrates upon essentially one problem only, namely the numerical computation of multiple integrals. Indeed eight of its ten chapters are devoted to it, and only the last two chapters discuss other topics (interpolation and differential and integral equations). Numerical integration is almost as old as integration itself. Some of the classical polynomial interpolation formulae are due to Newton. In fact, Newton himself (already in 1676; see, e.g., [10, p. 231]) used those formulae for the approximate computation of definite integrals. Better known and more accurate formulae for "mechanical quadrature", as numerical integration (especially in one variable) used to be called until quite recently (see, e.g., [10 and 16]), are those of Simpson, Weddle, Stirling, Bessel, Lagrange, and Gauss, among others. While originally devised only for simple integrals, they were soon used, by iteration, also for double integrals. Nevertheless, number theory played hardly any role in the most important problem related to these formulae, namely the estimation of the maximal error term; this was obtained by analytic methods.

The main idea of the quadrature formulae is to replace an integral $\int_a^b f(x)\,dx$ by a finite sum $\sum_{n=0}^{k} a_n f(x_n)$, $a = x_0 < x_1 < \cdots < x_k = b$, a_n independent of $f(x)$, in such a way that the size of the error term

$$R_k(f) = \left| \int_a^b f(x)\,dx - \sum_{n=0}^{k} a_n f(x_n) \right|$$

should be as small as possible, for any function $f(x)$ of a given class (such as functions of bounded variation, continuous functions, twice differentiable functions, etc.).

In many of the formulae, the points x_n are obtained by simply subdividing the interval $[a, b]$ into k equal parts. However, in Gauss' formula, normalized for the interval $[-1, +1]$, the x_n are the zeros of the kth Legendre polynomial. Gauss showed that, for the same amount of computation (as measured, e.g., by the number and precision of the terms used) one can improve considerably the result by a judicious choice of the points of subdivision.

This remark may be considered as the starting point of many future developments. Indeed, the question arises: can we select the points where we compute the function to be integrated in such a way that the average of very few such functional values, perhaps with proper weights, should yield the value of the integral over a unit interval with a minimal error? And next, can this procedure be used in a Euclidean space of arbitrarily many dimensions? This second question is far from trivial. Indeed, any integral in one variable, $\int_a^b f(x)\,dx$, can be normalized to an integral over the unit interval [0, 1] by a simple linear change of the variable. This, however, is no longer the case even in two dimensions; not every simply closed curve, even if convex, can be mapped onto the unit square by similarity transformations and rigid motions. The problem is still more difficult in higher dimensions and, in particular, for nonconvex volumes. Even in these cases, however, the very definition of the Riemann integral suggests that if we average the value of a reasonably smooth function at sufficiently many points with a sufficiently regular distribution over a unit volume of fairly arbitrary shape, we ought to obtain a good approximation to the integral of that function over the given volume.

With the advent of electronic computers it became possible to implement this approach. An element of randomness entered the choice of points, and that earned the method the name of Monte Carlo. The Monte Carlo method knew a number of successes, but soon also its limitations became apparent. It became clear once more that one could improve the result by a judicious choice of points. This was realized by Korobov (see [**11**, **12** and **13**]) and, independently and almost simultaneously, by Hlawka [**8**] somewhat over 20 years ago. These events signalled the start of the thorough use of number theory in numerical analysis. Shortly afterwards, again independently, a similar method was used by Conroy (see [**3**]) in the evaluation of a multiple integral that occurs in physical chemistry. Today this method, originally called by Korobov the *optimal coefficients method*, is usually refered to as the *good lattice points* (g.l.p.) method.

The leading ideas of the method are as follows: One attempts to find a good approximation to $\int_0^1 \cdots \int_0^1 f(\mathbf{x})\,d\mathbf{x}$ by a finite sum, as before. Here $\mathbf{x} = (x_1, \ldots, x_s)$ is a vector in s-dimensional Euclidean space and $d\mathbf{x} = dx_1 \cdots dx_s$. The accuracy of the procedure is much improved if one assumes that $f(\mathbf{x})$ is periodic, of period one, in each of its s variables. This means that it possesses a multiple Fourier series $f(\mathbf{x}) = \Sigma_{\mathbf{m}} C(\mathbf{m}) e^{2\pi i(\mathbf{x}\cdot\mathbf{m})}$, where $\mathbf{m}$ runs through all integral vectors in s-dimensional space and $(\mathbf{x}\cdot\mathbf{m}) = \sum_{i=1}^{s} x_i m_i$ is the inner product. Under these conditions we consider sums of the form $n^{-1}\sum_{r=1}^{n} f(r\mathbf{a}/n)$. Korobov and Hlawka have shown that it is possible to choose the vector $\mathbf{a} = \mathbf{a}(n)$ in such a way that for all functions $f(\mathbf{x})$ of a certain class,

$$\left| \int_0^1 \cdots \int_0^1 f(\mathbf{x})\,d\mathbf{x} - \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\mathbf{a}\right) \right| \leqslant Cn^{-\alpha}(\log n)^{\beta}.$$

Here C is an absolute constant (for the given class of functions) and α depends on the smoothness of those functions. The vector $\mathbf{a} = \mathbf{a}(n)$ has to be chosen by

some (not uniquely determined) method, so that the diophantine equation

$$(*) \qquad (\mathbf{a} \cdot \mathbf{m}) \equiv 0 \pmod{n}$$

should have no "small" solutions $\mathbf{m} = (m_1, \ldots, m_s)$. Specifically if we set $\overline{m}_i = \max(1, |m_i|)$, one requires that $\|\mathbf{m}\| = \prod_{i=1}^{s} \overline{m}_i$ should exceed a certain lower bound for all solutions $\mathbf{m}$ of $(*)$ (see, e.g., Lemma 3.9). Once we settle on a method for the selection of $\mathbf{a}$ as a function of n, we can, using the value of α, also compute β. It is clear that by taking n successively larger we can reduce the error below any preassigned limit, but for each new $n, \mathbf{a} = \mathbf{a}(n)$ has to be recomputed—and that is by far the most time-consuming part of the process. According to the method used to determine $\mathbf{a}$, the authors speak of *p-points*, *good points* (g.p.) and *good lattice points* (g.l.p.). For the purpose of this review we shall largely ignore these distinctions.

Although there exists a theorem to the effect that the measure of the set of good points is one, the effective construction of even a single one of them is not trivial (see Baker [**1** and Schmidt **17, 18**]). Korobov, Zaremba [**19, 21**], Halton [**7**] and others have suggested methods for the economical construction of g.l.p., but one of the most important contributions to this topic is due to the present authors (who, by the way, define g.l.p. in a way slightly different from their predecessors).

Once this problem is solved, one has to extend the results (a) to functions that fail to be periodic with period 1, and (b) to volumes other than s-dimensional cubes. At least three methods have been proposed to handle problem (a). The simplest is probably to set $F(x) = \frac{1}{2}(f(x) + f(1 - x))$. Then $F(0) = F(1)$ and $F(x)$ can be defined outside $[0, 1]$ by periodicity as a continuous function. It also is clear that $\int_0^1 f(x)\,dx = \int_0^1 F(x)\,dx$, and the previous method applies to the last integral. The generalization to s variables is, of course, immediate. Other methods that have been proposed to overcome the lack of periodicity are changes of variables and the use of Bernoulli polynomials (besides the present book also see, e.g., [**6**]). As for (b), the solutions suggested are perhaps not entirely satisfactory. An obvious approach is the introduction of the characteristic function for the set S, $\chi(\mathbf{x}) = 1$ if $\mathbf{x} \in S$, $\chi(\mathbf{x}) = 0$ otherwise.

Then, if $S \subset G_s$ ($G_s = s$-dimensional unit cube) we may apply previous methods to $F(\mathbf{x}) = \chi(\mathbf{x}) f(\mathbf{x})$ and integrate over G_s. Unfortunately, $\chi(\mathbf{x})$ (hence $F(\mathbf{x})$) is far from being smooth; in fact, it is not even continuous and, in general, $F(\mathbf{x})$ does not belong to the class of $f(\mathbf{x})$. One may either smooth out $F(\mathbf{x})$ or use other procedures. Here the concept of isotropic discrepancy (see [**20**]) plays an important role, but this cannot be discussed here.

The book by Hua and Wang handles these and related problems with greatest care. Perhaps the most remarkable feature of the book is the large amount of algebraic number theory presented and used. Of the ten chapters, more than half (Chapters 1, 2, 3, most of 4, half of 5, all of 6) are devoted to basic number theory.

The text starts with a thorough discussion of algebraic number fields and their units, followed by a study of certain symmetric functions. Next, the PV (Pisot-Vijayaraghavan) numbers are introduced (the algebraic integer α is a PV

number if $\alpha > 1$, but for all its conjugates, $|\alpha^{(i)}| < 1$). The uniform distribution modulo one is carefully presented (but H. Weyl's name barely appears in a note at the end of Chapter 3). A relevant theorem of Vinogradov is proven and different types of discrepancies are defined and compared. Here the work of van der Corput, Hammersley, Halton, Hlawka, Zaremba, Niederreiter, and W. Schmidt is acknowledged, together with that of Korobov, the authors, Khintchine, and Bahvalov.

Because of the great importance of the concept of the discrepancy of a set of points, we recall its definition. Let $P_n(k) = (x_1^{(n)}(k), \ldots, x_s^{(n)}(k))$, $1 \leqslant k \leqslant n$ be a set of n points in the s-dimensional unit cube G_s, denote by $\gamma = (\gamma_1, \ldots, \gamma_s)$ any fixed point of G_s and set $|\gamma| = \gamma_1\gamma_2 \cdots \gamma_s$. Let $N_n(\gamma)$ denote the number of points of $P_n(k)$ that satisfy the inequalities $0 \leqslant x_i^{(n)}(k) < \gamma_i$ for $1 \leqslant i \leqslant s$. Then

$$\sup_{\gamma \in G_s} \left| \frac{1}{n} N_n(\gamma) - |\gamma| \right| = D(n)$$

is called the *discrepancy* of the set of points $P_n(k)$. If we have an increasing sequence of integers n_i, then, we may compute $D(n_i)$ for each of these. If $\lim_{n \to \infty} D(n_i) = 0$, the sequence $P(k) = (x_1(k), \ldots, x_s(k))$, $k = 1, 2, \ldots$, is said to be *uniformly distributed* in G_s.

The text continues with the study of approximations by rationals and of the number of solutions of diophantine equations and systems.

This theoretical preparation is used (at the end of Chapter 4) to compute the discrepancies of several sets of points that are either g.p. or g.l.p. Here the author's own contributions play a prominent role (use of PV numbers and of generalized Fibonacci numbers, defined by $F_n = \sum_{h=1}^{s} F_{n-h}$ for $n \geqslant s$, $F_j = 0$ for $j = 0, 1, \ldots, s-2$, $F_{s-1} = 1$).

Problems of uniform distribution are considered and functions of bounded variation in the sense of Hardy and Krause (already considered in this context by Zaremba [**22**]) are defined and studied.

Let $P_n(k)$ $(1 \leqslant k \leqslant n)$ be any set of points with discrepancy $D(n)$. Then, if $f(\mathbf{x})$ is a function of bounded variation (henceforth always understood in the sense of Hardy and Krause) of total variation $V(f)$, where $f(\mathbf{x})$ is not necessarily periodic, it is shown that

$$\left| \int_{G_s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{n} \sum_{k=1}^{n} f(P_n(k)) \right| \leqslant V(f)\, D(n).$$

The proof of this fundamental inequality is far from trivial and is given in full detail.

There are two requirements that the points $P_n(k)$ have to satisfy in order to lead to useful quadrature formulae: (1) they have to have a low discrepancy (i.e., they should be distributed as regularly as possible); and (2) they should be easily computable.

The authors list some 15 quadrature formulae, each with an upper bound of its error. Also, given *any* $P_n(k)$ in G_s, the authors construct a function $f(\mathbf{x}) \in C^{\alpha}$ ($\alpha = q + \lambda$, i.e. $f(\mathbf{x})$ has continuous derivatives up to order q and

the last, no longer differentiable derivative satisfies a Lipschitz condition of order λ) for which

$$\left| \int_{G_s} f(\mathbf{x})\, d\mathbf{x} - \frac{1}{n} \sum_{k=1}^{n} f(P_n(k)) \right| > c(q, \lambda, s) n^{-\alpha/s}.$$

This shows that, regardless of how well the points $P_n(k)$ had been selected, there is a limit on the precision of these quadrature formulae, beyond which we cannot hope to improve them, at least as long as we impose upon the functions $f(\mathbf{x})$ only smoothness, without periodicity.

At this point, periodic functions are introduced and a norm $\| f^{\alpha} \|$ is defined. If $f(\mathbf{x}) \sim \Sigma_{\mathbf{m}} C(\mathbf{m}) e^{2\pi i(\mathbf{m}\cdot\mathbf{x})}$ and $| C(\mathbf{m}) | \leqslant C/\|\mathbf{m}\|^{\alpha}$, then $f(\mathbf{x})$ is said to belong to the class $E_s^{\alpha}(C)$. Two other classes of functions, $Q_s^{\alpha}(C)$ and $H_s^{\alpha}(C)$, are also defined. They satisfy the inclusion relations $H_s^{\alpha}(C) \subset Q_s^{\alpha}(C_2) \subset E_s^{\alpha}(2^s C_2)$, but we shall not define them here more precisely. For functions of these classes, more precise quadrature formulae can be obtained than for nonperiodic functions. Hence, there is interest in reducing nonperiodic functions to periodic ones, and the previously mentioned three methods for this reduction are presented. Finally, the numerical integration of periodic functions is presented for each of the three classes defined and for selections of the $P_n(k)$ as p, g.p., or g.l.p. By estimating lower bounds for the error terms, it is shown that often the upper bounds of the errors are of the right order of magnitude, so that in general (i.e., for particularly bad functions of their class), the error cannot be further improved. This leaves open the possibility (in fact, the likelihood) that for the great majority of the functions to be integrated, the error is much smaller than the conservative theoretical upper bound obtained.

A full chapter (8th) presents numerical work. Specifically, g.p. and g.l.p. are computed by several methods, and the errors bounds to which they lead are compared to each other and to the actual errors in specific numerical examples. In this work the generalized Fibonacci sequence is found particularly useful. Also estimates for the number of operations and computing time are made and are related to the obtainable accuracies. Frequent references are made to several papers published in Zaremba's collection [**23**], including at least one lengthy verbatim quotation. Several numerical tables in the appendix are explained and discussed. Also several conjectures are formulated to the effect that the results obtained are probably better than what we can actually prove by our present estimates of the worst case error terms. Finally, as already mentioned, the last two chapters discuss interpolation for previously defined classes of functions and the numerical solution of differential and integral equations (both of Fredholm and Volterra type) respectively.

There is no doubt about the value and usefulness of this book—presumably the only available complete and systematic presentation of this important and interesting material.

It is the more to be regretted that the reading of the book is rendered difficult and at times actually unpleasant by certain shortcomings that may easily have been avoided. The book has no index. The translation is occasionally awkward (see, e.g., p. 27, lines 3–5). The proofreading does not appear to

have been made with the necessary care; indeed, there are many printing errors and, while some are easily corrected by the reader, others may be quite confusing (e.g., the exponents r_i in Theorem 1.1 should be γ_i). Proper names are often misspelled (Minkowski, p. 33 last line; Hardy, p. 99, line 12; Korobov several times, occasionally (p. 86, line 9) as Kopobov, which suggests a translation from the Russian rather than from the Chinese). A section has the title "The Halton Theorem", but contains several theorems and Halton's is not identified among them. Some terms are used with a meaning different from their usual one (the sums of tth powers of the roots of a polynomial equation are called *elementary* symmetric functions). Often a symbol is not defined. Occasionally, it is easy to guess its meaning, like $\{x\}$ in Theorem 3.2 for the fractional part of x. Other times, like for $\langle x \rangle$ on p. 60, the reader may think first of the similar symbol defined in Theorem 1.5, where it means the group generated by x; this guess is wrong. The reader fortunate enough to know the book **[14]** by Kuipers and Niederreiter will realize that the symbol stands for the distance to the nearest integer. The reference to Theorem 7.4 (p. 155, line 6) should be to Theorem 7.14, etc. The examples could be multiplied. Some readers may wish to precede, or supplement, their reading of the Hua-Wang book by studying the papers by S. Haber, S. K. Zaremba, D. Maisonneuve, and H. Niederreiter in **[23]**, all of which are quoted in the present book. A knowledge of the book **[14]** by Kuipers and Niederreiter, although this is restricted to the problem of uniform distribution, may also help in the reading of the present book.

In spite of the mentioned superficial shortcomings, which should easily be taken care of in a new edition, the book by Hua and Wang is a most valuable contribution to numerical integration and to the solution of differential and integral equations. The book contains much material due to the authors themselves and, in many cases, the methods suggested have lead to the most accurate results with a minimum of computations. The tables of the Appendix are valuable by themselves.

Finally, the book itself is a brilliant illustration of the practical usefulness of pure, abstract number theory.

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Rational homotopy theory and differential forms, by Phillip A. Griffiths and John W. Morgan, Progress in Mathematics, vol. 6, Birkhäuser, Boston, Basel, Stuttgart, 1981, xi + 242 pp., $14.00.

Differential forms have motivated algebraic topology from its very beginning. Efforts to understand geometry arising from the Stokes formula have led to homology and cohomology theory. On the other hand, differential forms have always played an important role in the study of topological properties of Riemann surfaces.

For a long time, the only algebraic topological information of general validity provided by differential forms has been the de Rham theorem. Recently it has been discovered that differential forms furnish algebraic topological information extending far beyond the de Rham cohomology group in a systematic way.

There are two ways of obtaining additional algebraic topological results from differential forms: the method of iterated path integrals and that of minimal models. The former method takes advantage of the geometric wealth of the path space, which may be regarded as expressing the dynamics of the space in consideration. Usual differential forms are repeatedly integrated to