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Solvability of nonlinear equations and boundary value problems, by Svatopluk Fučík, Mathematics and its Applications, vol. 4, D. Reidel Publishing Company, Dordrecht, Holland/Boston, USA, London, England, 1980, 400 pp., $29.95.

Someone said it first: "Having a theory of nonlinear differential equations as opposed to a theory of linear differential equations is comparable to having a theory of nonchickens as opposed to a theory of chickens". (Professor Jean Mawhin has informed the reviewer that "bananas" is sometimes substituted for "chickens".) Since the book under review is mainly concerned with nonlinear differential equations and since it is impossible to review an area as broad as that of nonchickens, past, present, and future in only a few thousand words, the reviewer must restrict his attention to those types of nonlinear differential equations genuinely dealt with by the author. A more "global" discussion can be found in the review **[11]** of another book co-authored by Fučík.

In the reviewer's opinion a suitable subtitle of the book would be *A survey of the boundary value problem*

$$\Delta u + g(u) = h(x), \qquad u \mid \partial\Omega = 0 \tag{1}$$

and variations thereof. The most common variations considered are higher-order, semilinear, elliptic boundary value problems, ordinary differential equations with periodic boundary conditions, and periodic-boundary value problems for semilinear wave equations. A case may be made against this description on the basis that much of the book concerns the operator equation

$$Lu = Nu \tag{2}$$

where u is in a Banach space, L is linear, and N is nonlinear. However, those familiar with the book would probably agree that most of the theory developed for (2), by the author, was motivated by specific results for the problem (1) and not the other way around. In any case, this review is for the nonexpert and for this reason we emphasize the more concrete object (1).

Let us suppose, for simplicity, that g is at least of class C^1 and that h and the bounded domain $\Omega \subseteq \mathbf{R}^N$ ($N \geqslant 1$) are sufficiently regular. The generality of the nonlinearity makes it difficult to give a useful classification of problems of the form (1) but there are four types which stand out in the current literature. These types, which are emphasized in the book, are nonresonance problems, resonance problems, problems with rapid nonlinearities, and problems with jumping nonlinearities.

Although there does not seem to be any precise definition of what a nonresonance problem is, most of the persons who have used the term would probably agree that (1) is a nonresonance problem if $g(s)/s$ remains in a closed interval which does not contain any eigenvalues of the linear problem

$$(3) \qquad \Delta u + \lambda u = 0, \qquad u \mid \partial\Omega = 0$$

for $|s|$ sufficiently large. It appears that a special case of the condition was first considered in a classic paper of Hammerstein [32] on nonlinear integral equations. The results of this paper, applied to an integral equation, equivalent to (1), show that if $N \leqslant 2$ and $g(s)/s \leqslant \gamma < \lambda_1$ for $|s|$ sufficiently large, where λ_1 is the least eigenvalue of (3), then (1) has at least one solution for any smooth h. Using variational methods, Hammerstein was able to show more generally that if $G' = g$, γ is as above, and for some constant C the weaker condition $G(s) \leqslant \gamma s^2/2 + C$ holds for all s, then (1) is solvable; he also showed that if the stronger condition $g'(s) \leqslant \gamma$ holds for all s, then (1) has a unique solution. Dolph [23] extended the work of Hammerstein by showing that if I is a closed interval containing no eigenvalue of (3) and $g(s)/s \in I$ for $|s|$ large, then (1) is solvable; moreover, he showed that the stronger condition $g'(s) \in I$ for all s implies unique solvability. Very recently, Dancer [22] established the following complementary result: If the range of g' contains an eigenvalue of (3) in its interior, then there exists a smooth h for which (1) has more than one solution.

One type of nonresonance problem which has been investigated after Fučík's death in 1979 is the question of existence of nonzero solutions of (1) in the case where h is identically zero and $g(0) = 0$. For example, a recent abstract result obtained by Amann and Zehnder [3] shows that if $\lim_{s\to\infty} g'(s) = \lim_{s\to-\infty} g'(s) = g'(\infty)$, $g(0) = 0$, $g'(\infty)$ is not an eigenvalue of (3), and the half-open interval $(g'(0), g'(\infty)]$ contains an eigenvalue of (3), then $\Delta u + g(u) = 0$ has a nonzero solution. Chang [16] has given a short proof of their underlying abstract principle. Stronger multiplicity results have been obtained with more assumptions on g—see for example, Ambrosetti and Mancini [6] and Hempel [34] and Clark [18, 19] for the case g even.

In Fučík's book nonresonance problems are discussed in two brief chapters which comprise a part of the book entitled *Nonlinear perturbations of linear invertible operators*. As explained in the book this part serves mainly to introduce the part entitled *Nonlinear perturbations of linear noninvertible operators* which, in keeping with current terminology, concerns resonance problems.

The prototypic resonance problem is a problem of the form (1) in which $\lim_{s\to\infty} g(s)/s = \lim_{s\to-\infty} g(s)/s = \lambda_k$, where λ_k is an eigenvalue of (3). A large portion of the book is concerned with the special case where $g(s) = \lambda_k s + f(s)$ and $f(s)$ is bounded on the real line. As was shown in [37], if the limits $\lim_{s\to\pm\infty} f(s) = f(\pm\infty)$ exist, if λ_k is a simple eigenvalue of (3), and either of the conditions $f(-\infty) < f(s) < f(\infty)$ or $f(-\infty) > f(s) > f(\infty)$ hold for all s, then a necessary and sufficient condition for solvability of (1) can be given. For example, if f satisfies either of these conditions, $k = 1$, Θ_1 is an eigenfunction corresponding to λ_1, and $(\,,\,)$ denotes the usual $L^2(\Omega)$ inner product, then the necessary and sufficient condition is that $(h, \Theta_1)/(1, \Theta_1)$ belong to the range of f. Even if one or both of the limits $f(-\infty)$, $f(+\infty)$ are infinite, this condition is necessary and sufficient for solvability if $f(-\infty) > f(s) > f(\infty)$ for all $s \in (-\infty, \infty)$. This result is a special case of a theorem of McKenna and Rauch [**43**] which also applies to distributional solutions of higher-order, semilinear boundary value problems with strong nonlinearities, that is, nonlinearities which do not map $L^2(\Omega)$ into itself. The work of de Figueiredo and Gossez [**25**] also gives a unified treatment of some resonance and nonresonance problems for higher-order, semilinear problems with strong nonlinearities.

Recently, there have appeared several research papers (for example [**20**, **26**]) which have considered the case $g(s) = \lambda_k s + f(s)$ where $f(\pm\infty) = 0$ and have given sufficient conditions for solvability of (1) when $k = 1$. Since the methods of these papers rely strongly on the fact that an eigenfunction corresponding to λ_1 does not vanish in Ω, there do not seem to be any immediate extensions to the case $k > 1$.

Fučík's book contains discussions of several abstract theorems dealing with resonance problems which are mainly due to members of the Prague school [**27**, **29**, **44**]. The book also contains a section on resonance and nonresonance problems for periodic ordinary differential equations and one on the use of critical point theory in the study of resonance problems. The section on periodic differential equations is largely influenced by results of Mawhin and his co-workers (see, for example, [**8**, **31**, and **41**]) while the one on critical point theory extends the approach used in [**1**].

An important reference for resonance problems, which supplements the book, is Chapter II of the substantial paper by Brezis and Nirenberg [**12**]. This paper obtains many results concerning resonance as applications of theorems concerning the range of the sum of two nonlinear operators. The reviewer also recommends the recent monograph of Haraux [**33**] which discusses resonance problems for abstract evolution equations.

Following Fučík, (1) is said to be a problem with a rapid nonlinearity if $\lim_{s\to\pm\infty}(g(s)/s) = \infty$, e.g. $g(s) = s^3$; the solvability of (1) when $g(s) = -s^3$ follows from Hammerstein's paper if $N \leqslant 2$ and from the well-known method of subsolutions and supersolutions [**49**] for arbitrary N. In contrast, it appears to be unknown if $\Delta u + u^3 = h(x)$, $u \mid \partial\Omega = 0$ is solvable if $N = 2$.

Problems with rapid nonlinearities are better understood when $N = 1$. Using the "shooting method" Ehrmann [**24**], Fučík and Lovicar [**28**], and others have shown that if $N = 1$ and g is a rapid nonlinearity, then (1) has infinitely many solutions. Fučík and Lovicar were also able to prove the existence of at least one solution when the boundary conditions are periodic. Cesari [**14**] has given a functional-analytic proof of the solvability of (1) when $N = 1$, $g(s) = s^3$ and $h(x) = \sin x$ and he has treated similar problems with different boundary conditions in [**15**]. Variants of Cesari's method have also been used to investigate resonance problems (see [**27**, **29**, or **37**]).

Although the book does not discuss problems with rapid nonlinearities for $N > 1$, some progress has been made in this direction, especially for the case where $g(0) = 0$ and h is identically zero. For special forms of g there are results for such problems that follow from an old theorem of Lyusternik [**40**] concerning constrained critical points of even functionals on real Hilbert spaces. For example, if $N \leqslant 3$ an application of this theorem implies the existence, in a suitable Sobolev space, of a sequence of functions $\{u_m\}_{m=1}^{\infty}$ with norms equal to one and a corresponding sequence of positive numbers $\{\lambda_m\}_{m=1}^{\infty}$ such that $\Delta u_m + \lambda_m u_m^3 = 0$, $u_m \mid \partial\Omega = 0$, and $\lim_{m\to\infty} \lambda_m = \infty$. Setting $v_m = \lambda_m^{1/2} u_m$ one obtains an unbounded sequence of solutions of $\Delta u + u^3 = 0$, $u \mid \partial\Omega = 0$. This is also implied by a rather general result obtained by Ambrosetti and Rabinowitz in **[5]**. They have shown that if g is odd, $g(s)$ does not grow fast as $s^{(N+2)/(N-2)}$ as $|s| \to \infty$, and there exists a constant $\Theta > 2$ such that $0 < G(s) \leqslant \Theta^{-1} s g(s)$ for $|s|$ large, where $G' = g$, $G(0) = 0$, then (1) has solutions with arbitrarily large norms when h is identically zero.

If g is not odd but satisfies the extra conditions $g(0) = 0$ and $g'(0) < \lambda_1$, then the "mountain pass theorem" can be used to ascertain the existence of at least one nonzero solution of $\Delta u + g(u) = 0$, $u \mid \partial\Omega = 0$. Benci and Rabinowitz **[9]** have obtained other conditions for the existence of a nontrivial solution when g is not necessarily odd. Very recently and simultaneously, Bahri and Berestycki **[7]** and Struwe **[51]** have shown that if $g(s)$ is a rapid nonlinearity which satisfies certain technical assumptions and which does not grow faster than a certain power depending on N, then the inhomogeneous problem (1) has infinitely many solutions.

On p. 279 of the book under review Fučík poses the question of whether or not the problem $\Delta u + |u|^{\varepsilon} u = h(x)$, $u \mid \partial\Omega = 0$ is solvable if $\varepsilon > 0$ is sufficiently small. When $N = 3$ the result of Bahri, Berestycki, and Struwe answers the question in the affirmative if $\varepsilon < 0.693$.

The term "jumping nonlinearity" was first used by Fučík in **[30]** although it appears that Ambrosetti and Prodi **[4]** wrote the first paper on the subject. In the current literature (1) is usually called a problem with a jumping nonlinearity if it is not a nonresonance problem, as defined above, and

$$\limsup_{s\to -\infty} g(s)/s < \liminf_{s\to +\infty} g(s)/s.$$

Ambrosetti and Prodi showed that if g is strictly convex and the limits $g'(\pm\infty)$ satisfy $0 < g'(-\infty) < \lambda_1 < g'(\infty) < \lambda_2$ then, under standard regularity assumptions, there exists a closed, connected manifold M in the space $C^{\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ of codimension one such that the complement of M consists of two open, disjoint, connected sets U_0 and U_2 having the properties that (1) has zero, exactly one, or exactly two solutions depending on whether h belongs to U_0, M, or U_2, respectively. Subsequent work of Kazdan and Warner **[36]** and Berger and Podalak **[10]** elucidated the form of U_0, M_1 and U_2. Results of the first of these papers showed that if $\lim\sup_{s\to-\infty} g(s)/s < \lambda_1 < \liminf_{s\to+\infty} g(s)/s$ and $h(x) = h_1(x) + s\Theta(x)$, where Θ is a positive normalized eigenfunction corresponding to λ_1 and h_1 is orthogonal to Θ, then there exists a number s_0 depending on h_1 such that

$$\Delta u + g(u) = h_1(x) + s\Theta(x), \qquad u \mid \partial\Omega = 0 \tag{4}$$

is solvable if $s > s_0$ and is not solvable if $s < s_0$.

Fučík's book discusses extensions of the Ambrosetti-Prodi theorem through the work of Kazdan and Warner, but many other papers on this subject have appeared in the last few years. A wide-open problem concerns the multiplicity of solutions of (4). In this direction it was simultaneously shown by Amann and Hess [2] and Dancer [21] that if $g(s)$ satisfies the above-mentioned conditions and satisfies a suitable growth condition as $s \to \infty$, then (4) is solvable for $s = s_0$ and has at least two solutions for $s > s_0$. McKenna and the reviewer [38] then showed that the additional hypotheses that λ_2 is simple and $\lim_{s\to\infty} g(s)/s \in (\lambda_2, \lambda_3)$ imply the existence of at least three solutions for sufficiently large s. Next, Solimine [50] showed that this is still true under the weaker conditions $\limsup_{s\to-\infty} g(s)/s < \lambda_1$, $\gamma = \lim_{s\to\infty} g(s)/s > \lambda_2$, and γ is not an eigenvalue of (3), and, very recently, Hofer [35] showed these same conditions imply the existence of at least four solutions for large s.

The above-mentioned results of Dancer, Amann, Hess, and Hofer prove the first two cases of the conjecture made in [38] that if $\limsup_{s\to-\infty} g(s)/s < \lambda_1$, $\lambda_m < \lambda_{m+1}$ for some $m \geq 1$, where each eigenvalue is counted as often as its multiplicity, and $\lambda_m < \lim_{s\to\infty} g(s)/s < \lambda_{m+1}$, then (4) has at least $2m$ solutions for sufficiently large s. In [39] it is shown that this is true if $N = 1$.

Problems with jumping nonlinearities in which the interval

$$\left(\limsup_{s\to-\infty} g(s)/s,\ \liminf_{s\to+\infty} g(s)/s\right)$$

does not contain λ_1 were apparently considered in the same year, for the first time, by both Fučík [30] and Podolak [46]. Fučík considered an ordinary differential equation in which the above-mentioned interval may contain more than one eigenvalue while Podolak showed that the Ambrosetti-Prodi phenomenon takes place if $g(s) = \lambda_k s + f(s)$, $k \geq 2$, λ_k is simple, $\lim_{s\to\infty} f'(s) = \varepsilon$, $\lim_{s\to-\infty} f'(s) = -\varepsilon$ where $\varepsilon > 0$ is small, f is convex, and $(\phi_k, |\phi_k|) \neq 0$ if ϕ_k is an eigenfunction corresponding to λ_k. In this book, Fučík discusses results concerning this problem, which is still a hot research topic, up to 1978. A recent, noteworthy contribution, which extends the work of Podolak, is that of Ruf [48].

To do justice to the methods which have been applied to the types of problems we have discussed would require us to at least double the size of this review; therefore, the reviewer will make only a few remarks concerning methods—a good discussion of modern methods in the theory of nonlinear boundary value problems can be found in the survey article [45] and a historical discussion can be found in the introductory part of [31].

What the reviewer finds fascinating about the study of (1) is that there are results involving diverse hypotheses on g which are obtainable by one particular method and there are results concerning very particular g which are obtainable by a combination of diverse methods. For example, the method of obtaining critical points of the inf-max type via a deformation argument has been used as the main tool to obtain results on resonance problems in [47], nonresonance problems in [19], problems with rapid nonlinearities in [5], and a problem with a jumping nonlinearity in [13]. On the other hand, a forthcoming paper by Mawhin and Willem [42] which is concerned with the particular resonance-type problem, combines order methods, degree-theoretic methods, variational methods, and convex analysis.

The methods most favored by Fučík in the book usually involve formulating a problem as an abstract equation of the form (2) and considering a splitting of

(2) into a pair of coupled equations involving variables from certain direct summands of the range and domain spaces of L. The coupled equations are then studied using degree theory, monotone operator theory, implicit function theorems, etc. This splitting technique is sometimes referred to as the alternative method (after Cesari [**14**, **15**]) and sometimes as the Lyapunov-Schmidt method as in [**48**]. The distinction between these two methods, as made in Chapter 2 of [**17**], is that in the Lyapunov-Schmidt method one of the direct summands in the domain space of L is the kernel of L, while in the alternative method, one of direct summands in the domain space may be any subspace containing the kernel.

In the study of (1) the splitting method has been effective except in the study of problems involving rapid nonlinearities where variational methods seem to be the only known effective tool when $N > 1$.

For the neophyte in nonlinear analysis the book is a good place to begin since the author presupposes little on the part of the reader and gently introduces each new topic with a discussion of a one-dimensional problem of the form (1).

An early version of the book consisted of mimeographed notes which were mailed throughout the world by the author. These notes stimulated some research which is reported on in the present version; this version should continue to stimulate research for quite some time to come.

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Applications of number theory to numerical analysis, by Loo Keng Hua and Yuan Wang, Springer-Verlag, Berlin, Science Press, Beijing, 1981, 241 pp., $39.00.

What is number theory good for? No one doubts that many branches of mathematics owe their existence to—or, at least, were strongly stimulated by—problems of the "real world", like those of physics, engineering, etc. Familiar examples are the calculus and the theory of differential equations needed in celestial mechanics; partial differential equations that are indispensable in hydrodynamics and so on. But number theory? Often number theorists, when challenged by our first question (usually asked by nonmathematicians) feel obligated to convince the questioner that number theory also can be useful. Sometimes its applications in problems of crystallography and, more recently, in cryptography are mentioned. Why it should be necessary to point out a "usefulness" in the commonly understood sense for number theory is something of a mystery to this reviewer. It appears quite certain that Diophantus, or Fermat, or Gauss studied this field of human knowledge because of its intrinsic interest and its peculiar beauty—and they really did not care one way, or the other, whether their elegant theorems would, or would not have "useful" applications.

Be that as it may, it turns out that like so many other branches of mathematics, developed by the "purest" of mathematicians, also number