References

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.
2. M. F. Atiyah, *K-theory*, Benjamin, New York, 1967.
3. ______, Topology **1** (1961), 125–132.
4. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 7–38.
5. J. C. Becker, *The span of spherical space forms*, Amer. J. Math. **94** (1972), 991–1025.
6. R. Bott, *The stable homotopy of the classical groups*, Ann. of Math. (2) **75** (1953), 409–448.
7. H. Hopf, *Über die abbildungen der dreidimensinalen sphäre auf die Kugelfläche*, Math. Ann. **104** (1931), 637–665.
8. M. Karoubi, *K-theory*, Grundlehren der. Math. Wiss., no. 226, Springer-Verlag, Berlin and New York, 1978.
9. C. B. Thomas, *Free actions by finite groups on* S^3, Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I., 1978, pp. 125–130.

VICTOR SNAITH

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 1, January 1983

0273-0979/82/0000-0779/$03.50

Finite simple groups, an introduction to their classification, by Daniel Gorenstein, Plenum Publishing Corporation, New York, 1982, x + 333 pp., $29.50.

After an effort of about 30 years involving more than a hundred mathematicians in several countries, the finite simple groups have apparently been classified. I use the work "apparently" because this achievement is a unique sociological phenomenon in the history of mathematics. Proofs of the many relevant results are scattered throughout the mathematical literature in 300–500 papers covering 5,000–10,000 pages. No individual has gone through the whole proof and checked all the details. This is not an entirely satisfactory situation. However most experts are convinced that the proof is essentially correct; any errors which occur are expected to be minor oversights or local errors which can be corrected by the methods that have been developed in the process of completing the classification. More importantly, no error is expected to change the end result, i.e. to lead to new simple groups.

Except at the level of foundations, mathematics is not a matter of faith, so it is not surprising that the announcement of the classification has been treated with some scepticism among mathematicians. Nevertheless it would be pointless (and probably impossible) for anyone at present to attempt to go through a complete proof and to check all the details, because the proof is continually being revised, simplified and shortened. This process has been dubbed revisionism. In part, such simplifications are due to the inevitable redundancy which occurs in an undertaking of this magnitude. However great successes have already been achieved for less obvious reasons. For example, the classification of simple groups with dihedral Sylow 2-groups originally took 221 journal pages [**7**, **8**], there is now a proof in 29 pages [**1**, **2**]. Such a simplification and others of a similar sort are due to the introduction of new ideas and to a deeper understanding of the structure of finite simple groups. In these circumstances brevity and clarity go hand in hand. It is not unreasonable to expect that perhaps in a decade or so it may be possible to write a book of normal length which will contain a complete proof of the classification and only assume prerequisites which are generally acceptable to mathematicians.

At present the proofs of many results needed for the classification of finite simple groups are notoriously long and technical. As often happens in mathematics, there are basic ideas and concepts, and there are applications of these. The basic ideas are frequently not difficult to understand, though sometimes their very formulation requires familiarity with many technical results. Unfortunately such ideas can only be justified by their successful application in proving the desired results. Here is the danger of being immersed in a sea of technicalities. Another reason for the excessive length of some papers is the need to handle many separate cases. Some of this case analysis will certainly be streamlined or eliminated in the near future; however, some of it is inevitable as it leads to true exceptions, which in turn are explained by the existence of sporadic groups. Indeed several of the sporadic groups first arose in this way.

Of course one cannot expect a proof of a result to be shorter than the statement. In the classification the statement entails a description of all finite simple groups. These in particular include the finite groups of Lie type whose existence and description is in itself quite complicated and rests on the theory of algebraic groups. Incidentally, the classification of finite simple groups amongst other things incorporates the classification of simple Lie groups and that of finite reflection groups.

Then there is the question of intuition. It appears to be a fact of nature that in classification theorems the end justifies the means. Intuition comes after the result rather than before. This is perhaps due to the fact that a classification theorem frequently includes the discovery and construction of at least some of the objects which are being classified. For example, exceptional Lie algebras first arose in this way much to Killing's surprise, see [**3**, p. 156]. But after all, this is precisely the reason why classification theorems are important!

Considering the morass of technical difficulties and the multitude of cases it is surprising that the classification was achieved in a relatively short time. Systematic work on the classification of finite simple groups began in the 50s, though R. Brauer had been a lonely pioneer before this. Such work, of course, built on earlier work in group theory, but was definitely a new beginning based on important and profound new ideas.

It is difficult to say what prompted this rebirth. The 50s were a decade in which abstraction in mathematics, in particular abstract algebra, reigned supreme, especially in the United States. (See [**10**] for an interesting historical discussion.) For nearly a century one of the central thrusts of mathematics had been the attempt to understand and use the infinite. During the 50s a major preoccupation of many mathematicians was the development of general theories and infinite constructions. Many great successes were achieved. Finite group theory was contrary to the spirit of the times and many mathematicians felt that questions about finite objects could not be either interesting or important.

The transformation of the theory of finite groups from a subject that was barely alive to one that a decade later was filling up journals with lengthy papers caught the mathematical community by surprise. This may partly account for the myths that seem to have built up about the theory of finite groups. I have the impression that mathematicians sometimes react to the subject somewhat in the same way that nonmathematicians react to mathematics. On the one hand, there seems to be a black box which mysteriously produces remarkable results. On the other hand there is the opposite point of view; this is an old subject and an old question, and if someone had only had the patience they could have classified the finite simple groups half a century ago. Human beings, even mathematicians, being occasionally inconsistent, both points of view are sometimes held by the same person.

This book should help to dispel such myths. It is aimed at a general mathematical audience. Every relevant concept is defined as it arises beginning with "simple group" on page 1, line 1. It is not a mathematics book in the usual sense. Many theorems are stated without proof, though frequently an attempt is made to indicate why a result is plausible. It also contains historical remarks and personal viewpoints of the author. For instance, §1.3 is entitled "Why the extreme length?"

Most of Chapter 1 should be easy to read, but §§1.4 and 1.5 contain many definitions. Much of this material will probably be familiar to most readers but the density of definitions may be a bit overwhelming the first time around.

Chapters 2 and 3 contain material which does not depend on all the definitions in Chapter 1. Chapter 2 consists of a description of all the finite simple groups concluding with a table. It is a pity that the author did not list exceptions or exceptional isomorphisms. See e.g. [**11**, pp. 213–215]. (To this latter list should be added the fact that for $l > 2$ and all n, $B_l(2^n) \simeq C_l(2^n)$ or equivalently $\mathrm{SO}_{2l+1}(2^n) \simeq \mathrm{Sp}_{2l}(2^n)$.) Chapter 3, entitled "Recognition theorems", contains properties of simple groups which lead to their characterizations.

The fourth and final chapter, which is also the longest, entitled "General techniques of local analysis", is somewhat more formidable. Here the definitions from Chapter 1 are needed and the reader can begin to see the types of arguments that are required. Much of this material was first developed for the classification, but many of the ideas and results presented here have already begun to find their way into text books on group theory and will become standard equipment for future generations of mathematicians.

The classification of finite simple groups, including the discovery and construction of the 26 sporadic groups, is a mathematical result of permanent interest. Quite independently of the methods used, a result of this depth must have consequences. The author discusses some of these very briefly in §1.7. (He is a bit overoptimistic here; the Alperin-McKay conjectures do not as yet follow from the classification. Theorem 1.52 must still be read as Conjecture 1.52.) The classification can act almost as a deus ex machina on occasion. To give an example chosen more or less at random consider the following.

Let $f(n)$ be the number of pairwise nonisomorphic simple groups of order n. Before the classification it was not known whether $f(n)$ is bounded. The classification yields that $f(n) \leqslant 2$. For $l > 2$ and odd q, the groups $PSp_{2l}(q)$ and $SO'_{2l+1}(q)$ are nonisomorphic simple groups of the same order. The only other pair consists of $A_8 \simeq SL_4(2)$ and $PSL_3(4)$ which have order $\frac{1}{2}(8!) =$ 20,160.

There are many questions in group theory and related fields which can be reduced to questions about simple groups. In practice this usually means that one has to handle the finite groups of Lie type, since by the classification they constitute the bulk of the simple groups. Sometimes such questions can be answered by routine checks, but sometimes their answer involves the theory of algebraic groups which thus becomes available for questions concerning arbitrary finite groups.

One particularly fruitful consequence of the classification is the complete description of all doubly transitive permutation groups. In view of the classification, this follows easily from [**5**] where the groups of Lie type are handled. The knowledge of all doubly transitive permutation groups can be used to answer innumerable old questions about permutation groups. More unexpectedly it also has consequences in number theory and logic, see e.g. [**3**, **6**]. It also of course limits possible codes, finite geometries and other finite combinatorial objects which admit doubly transitive automorphism groups.

The discovery and construction of the sporadic groups, especially F_1, the largest of these, is the aspect of the classification that has received most publicity both among mathematicians and others. The group F_1 had been dubbed "the monster" and is now trying to shed this image and be known as "the friendly giant". It appears that at present the most striking way in which finite simple groups may interact with other branches of mathematics is related to F_1. This is based on an observation of J. McKay which has led to a great deal of numerical evidence that points to possible connections between F_1, modular functions and infinite-dimensional graded vector spaces, perhaps Kac-Moody algebras, see [**4**].

The existence and uniqueness of the 26 sporadic groups has been established. However to get a better understanding of these groups it would be desirable to find an "explanation" of why they exist. In some cases the situation is fairly satisfactory.

The Steiner systems $S(5,6,12)$ and $S(5,8,24)$ are remarkable combinatorial configurations unlike any others. Their automorphism groups are the Mathieu groups M_{12} and M_{24}. These are the only 5-transitive permutation groups other than symmetric and alternating groups: (a fact long conjectured but only proved as a consequence of the classification). The Leech lattice is a blown up version of $S(5,8,24)$. It is the unique even unimodular lattice in 24 dimensions with no vectors of weight 2. This uniqueness is an essential reason why it is a geometric object of fundamental importance. The automorphism group Co.O of the Leech lattice involves about half of the sporadic groups and generally it is felt that these are well understood.

It would be desirable to find another geometric object, unique in some sense, presumably related to the Leech lattice, whose automorphism group is closely connected to F_1. At present most attempts to find such an object are intimately related to the numerical data mentioned above.

The group F_1 involves most, but not all, of the sporadic groups. Even if a geometric object connected with F_1 were to be discovered, it would still leave a few of the sporadic groups to be explained geometrically. It may be extremely difficult to find a suitable geometric object. It has been more than 80 years since the simple Lie groups were classified and there is still no really satisfactory geometrical description of $E_8(\mathbf{C})$.

I have here digressed slightly, as the questions about F_1 and its connections with other branches of mathematics are not considered in the book. These remarks are however evidence of the vitality of the theory of finite groups and the fact that this theory will impinge on several other areas in mathematics.

There is no reason for most mathematicians to attempt to understand the proof of the classification in detail. This book provides an opportunity for the interested reader to see some of the relevant methods and also to appreciate some of the difficulties that need to be surmounted. Even as revisionism transforms the proof, it does not appear as though the general outline of the proof will change very much (though, of course, it is impossible to predict the impact of new ideas) so that this book should remain relevant for some time to come. However, it must be realized that by the end of this book the reader will have seen only a rather superficial outline of the classification; two more volumes are promised on page 7. The book does contain enough material to provide a thorough understanding of the statement of the classification, which is in itself highly nontrivial.

The author is one of a handful of people who are familiar with all the strands that come together in the classification and are capable of writing a book such as this. The fact that he has done so puts the mathematical community in his debt. I don't think that this book will be replaced for some time to come. I recommend it to anyone who has the slightest curiosity about the classification of finite simple groups.

REFERENCES

1. H. Bender, *Finite groups with dihedral Sylow* 2-*subgroups*, J. Algebra **70** (1981), 216–228.

2. H. Bender and G. Glauberman, *Characters of finite groups with dihedral Sylow* 2-*subgroups*, J. Algebra **70** (1981), 200–215.

3. G. Cherlin, L. Harrington and A. Lachlan, $\aleph_0$-*categorical*, $\aleph_0$-*stable structures* (preprint).

4. J. Conway and S. Norton, *Monstrous moonshine*, Bull. London Math. Soc. **11** (1979), 308–339.

5. C. W. Curtis, W. M. Kantor and G. M. Seitz, *The 2-transitive permutation representations of the finite Chevalley groups*, Trans. Amer. Math. Soc. **218** (1976), 1–59.

6. W. Feit, *Some consequences of the classification of finite simple groups*, Proc. Sympos. Pure Math., vol. 37, Amer. Math. Soc., Providence, R. I., 1980, pp. 175–182.

7. D. Gorenstein and J. H. Walter, *On finite groups with dihedral Sylow* 2-*subgroups*, Illinois J. Math. **6** (1962), 553–593.

8. ______, *The characterization of finite groups with dihedral Sylow* 2-*subgroups*. I, II, III, J. Algebra **2** (1965), 85–151, 218–270, 334–393.

9. T. Hawkins, *Wilhelm Killing and the structure of Lie algebras*, Arch. for History of Exact Sciences **26** (1982), 127–192.

10. S. Mac Lane, *History of abstract algebra*, Texas Tech Univ. Math. Series **13** (1981), 3–35.

11. J. Tits, *Groupes simples et géometries associées*, Proc. Internat. Congr. Math. (Stockholm, 1962), Mittag-Leffler, Djursholm, Sweden, 1962, pp. 197–221.

WALTER FEIT