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WOLFGANG M. SCHMIDT

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Moduln und Ringe, by Friedrich Kasch, B. G. Teubner, Stuttgart, 1977, 328 pp., DM 52

Even though the concept of a ring was not formulated until the beginning of this century, rings had already been studied in the nineteenth century in the special cases of rings of algebraic integers, polynomial rings, power series rings and finite dimensional algebras over the real and complex numbers. Modules over rings are generalizations of vector spaces over fields, and were first studied by Dedekind and Kronecker over rings of algebraic integers and polynomials rings, in particular, in the special case of ideals.

The theory of rings and modules has in this century developed in various

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[1](#2) etc. signifies the position in the present collection of papers.

[2]We use the abbreviations K.V.S.S. for Kra. Vidensk, Selsk. Skrifter. I. Mat. Nat. Kl.

directions, under the influence of areas such as number theory, algebraic geometry and group representation theory. An important motivation has also been to get rid of unnecessary hypotheses to be able to formulate results in their greatest possible generality. For example many of the results on finite dimensional algebras over a field turned out to be independent of having a field present, and to hold more generally for the rings satisfying the descending chain conditions for left and right ideals, which are now called artin rings. This was the case for the classical Wedderburn-Artin structure theorem, stating that an artin ring modulo its radical is a finite product of matrix rings over division rings. And E. Noether was able to formulate many of the results for polynomial rings and rings of algebraic integers in the more general setting of commutative noetherian rings, that is, commutative rings satisfying the ascending chain condition on ideals. In the last decades there has been a strong influence of homological algebra and category theory, areas which arose towards the middle of this century in connection with work in topology. These areas have provided new methods for solving problems and have suggested new directions in the study of rings and modules, emphasizing more the modules over a ring rather than the ring itself. In particular, concepts such as projective and injective modules, generators and cogenerators have been put in the foreground.

The title of the book under review is *Moduln und Ringe*, but of course no attempt is made to give a comprehensive account of the whole theory of rings and modules. The central topic of the book is the class of artin rings called Quasi-Frobenius, or short, Q.F. rings. With the original motivation from group rings, the further study of these rings and their generalizations has a strong homological flavour.

The link between the theory of representations of a finite group G in a field k and the theory of artin rings is given by the observation, first made by E. Noether, that the representations of G in k correspond to the modules over the group algebra kG, which is an artin ring. This connection has determined directions of research in the theory of artin rings, and has at the same time provided a useful point of view in the theory of group representations. Many of the essential properties of the representations of a finite group G in a field k depend on the group ring $R = kG$ having the property that there is a k-linear homomorphism φ: $R \to k$ such that the kernel of R contains no nonzero left or right ideals. This condition is, for a finite dimensional k-algebra R, equivalent to the property that R and $\mathrm{Hom}_k(R, k)$ are isomorphic as left R-modules. Finite dimensional k-algebras R with this property are called Frobenius algebras, and the Q.F. algebras are the finite dimensional k-algebras R satisfying the somewhat weaker condition that each direct summand of the left R-module R is isomorphic to a summand of $\mathrm{Hom}_k(R, k)$ and conversely. These algebras were investigated in Nakayama's fundamental papers on the subject [8]. Amongst other things he showed that a finite dimensional k-algebra R is Q.F. if and only if every left and every right ideal I is equal to its double annihilator, that is $\mathrm{ann}_R(\mathrm{ann}_R\, I) = I$. Hall had earlier proved that a semisimple ring satisfies the above double annihilator condition. Getting rid of the underlying field, Nakayama also extended the definition of Frobenius and Q.F. algebras to Frobenius and Q.F. rings. We just mention that a Q.F. ring is an artin ring whose left and right ideals satisfy the double annihilator condition.

In the further development of this theory there has been a strong influence of homological algebra and category theory, and the Q.F. rings were characterized as those artin (or noetherian) rings R where R is an injective left R-module, and as those artin (or noetherian) rings R such that R as a left R-module is a cogenerator in the category of R-modules, that is, every left R-module is a submodule of a direct product of copies of R. These results indicate obvious ways of generalizing Q.F. rings beyond the artin case, and such and other generalizations of Q.F. rings have been investigated by several authors. It is also interesting to note that the Q.F. rings can be described completely using projective and injective modules, without assuming any chain conditions, namely as those rings R where each injective left R-module is projective [6] or as those rings R where each projective left R-module is injective [4].

The Q.F. rings also play a central role in connection with duality. The classical duality from the category of finite dimensional vector spaces over a field k to itself which sends a finite dimensional k-vector space V to $V^* = \mathrm{Hom}_k(V, k)$, has been of importance in linear algebra. Here the natural map from V to V^{**} is an isomorphism when V is finite dimensional. For a finite dimensional k-algebra R the above duality induces a duality between the categories of finitely generated left and right R-modules (using that for a left R-module V, $\mathrm{Hom}_k(V, k)$ is in a natural way a right R-module). This duality provides a useful tool in the study of finite dimensional k-algebras. Now there is another way of attempting to define a duality between module categories motivated by the one for vector spaces, by sending the left R-module V to $V^* = \mathrm{Hom}_R(V, R)$, which has a natural structure as right R-module. However, this way we do not in general, not even for a finite dimensional algebra R over a field k, get a duality between the finitely generated left and right R-modules, nor do we get that the natural map $V \to V^{**}$ is an isomorphism for a finitely generated R-module V. The rings such that the last property holds are called rings with perfect duality, and a ring with perfect duality is artin if and only if it is Q.F. Dieudonné studied the Q.F. rings from this point of view, which made it possible to simplify and extend earlier results [3]. Also Morita and Tachikawa made similar investigations, and Morita studied more generally the question when there exists some kind of duality between subcategories of module categories [7]. It should also be pointed out that the Q.F. rings are exactly the rings where the functor $\mathrm{Hom}_R(\ , R)$ induces a duality between the finitely generated left and right R-modules.

The last two chapters of the book under review deal with Frobenius and Q.F. algebras and rings and rings with perfect duality. In connection with the treatment of rings with perfect duality, where several characterizations of these rings are given, I think it would have been helpful for a novice in the field if an outline had been given of the more general results known on duality between module categories, in particular, on Morita duality. Also I think it should have been mentioned that the Q.F. rings are those rings where $\mathrm{Hom}_R(\ , R)$ induces a duality between the categories of finitely generated left and right R-modules, especially since from reading the introductory remarks to the chapter on perfect duality the reader might be led to believe that the rings with perfect duality are the rings where $\mathrm{Hom}_R(\ ,R)$ induces such a duality. Even though it may be an advantage for the reader who meets the subject for the first time not to have to worry about all kinds of generalizations an indication of the main results known, with references, would make it easier for him to find his way through the literature. Now it should be said that there is a list of papers on related material at the end of the book.

This book also discusses the theory of perfect rings and semiperfect rings and modules. The connection with the other topics in the book is given by the fact that a ring with perfect duality has to be semiperfect.

The perfect and semiperfect rings are examples of classes of rings which arose under the influence of homological algebra. They were introduced by Bass in his famous paper [2], where he defined a ring R to be left perfect if every left R-module has a projective cover and (left) semiperfect if every cyclic left R-module has a projective cover. Bass gave a series of interesting characterizations of these rings, in particular, he established a connection with chain conditions by showing that a ring R is left perfect if and only if R satisfies the descending chain condition on principal right ideals. In particular, an artin ring is left perfect, and the left perfect rings give the right setting for many of the essential properties of artin rings.

The book under review proves most of Bass' characterizations of left perfect rings, except those involving higher projective dimension and direct limits, as these topics are not treated in the book. The proofs are mostly essentially those of Bass, apart from the fact that the theory of semiperfect modules as studied by Mares and the author replaces some other arguments.

The first ten chapters of the book concentrate, after giving the very basic theory for rings and modules, on treating the topics necessary to give a selfcontained (except for some set theoretical results) treatment of the theory of the last three chapters. Hence injective and projective modules, generators and cogenerators, artin, noetherian and semisimple rings, radical and socle are among the topics given a broad discussion. These chapters also serve well as an introduction to other areas of ring and module theory with a strong homological flavour.

The book contains almost no misprints and is very well written. To mention an example, I found the exposition particularly nice in connection with the discussion of injective envelopes and projective covers, where the author manages to provide a good insight into the understanding of the fact that injective envelopes always exist, whereas projective covers do not.

Altogether, the book under review should serve well as text for a graduate course centering around the theory of Q.F. rings and also as a good start for somebody wanting to get into research in the areas treated in the book. Even though both Q.F. rings and perfect rings are treated in the recent book of Faith [5], this latter book treats several other main topics, so that it is still handy to have a smaller book if the topics which are treated coincide with the main interests of the reader. There is another recent book of about the same size and level as the book under review, by Andersen and Fuller [1], discussing basically the same preliminary topics and also perfect and semiperfect rings. However, this latter book does not discuss Q.F. rings, but gives instead a broad discussion of Morita equivalence and Morita duality. So in spite of a large overlap, the aims of the books are still different.

I close the review with expressing the hope that the fact that the book under review is written in German won't frighten too many prospective readers away.

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IDUN REITEN

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Hopf spaces, by Alexander Zabrodsky, Mathematics Studies, No. 22, North-Holland, Amsterdam, 1976, x + 223 pp., $18.50.

The subject of H-spaces is generally agreed to have begun in 1941 with the publication of Hopf's paper [5]. In the proceedings of the 1970 Neuchâtel conference on H-spaces, James [8] listed 347 entries for a bibliography on H-spaces. Since that time numerous articles on H-spaces have been published. It is therefore somewhat surprising that Zabrodsky's monograph is only the second book to appear which deals with H-spaces in general. Before discussing the book, I would like to provide some background on the subject itself.

It is quite easy to define the basic concept. An H-space (or Hopf space) consists of a topological space X with chosen point $* \in X$ and a continuous function μ: $X \times X \to X$ called the multiplication or H-structure on X. The requirement is that $*$ be a two-sided unit up to homotopy, that is, the maps $x \to \mu(x, *)$, $x \to \mu(*, x)$, and the identity map of X are all to be homotopic. If in the definition we replace homotopy by equality and write $\mu(x, y)$ as $x \cdot y$, we obtain $x \cdot * = * \cdot x = x$. The multiplication is then called strict and we shall refer to the resulting object as a topological quasi-group, a precursor of a topological group. Two important classes of examples of H-spaces are topological groups and the space of loops ΩY of an arbitrary space Y. The latter consists of continuous paths in Y parametrized by [0, 1] which begin and end at a fixed point of Y with multiplication of paths the same as in the definition of the fundamental group. H-spaces are studied because they are a natural object in homotopy theory and because they are a unifying concept