BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

PHASE TRANSITIONS IN $P(\phi)_2$ QUANTUM FIELDS

BY J. GLIMM,[1] A. JAFFE[2] AND T. SPENCER[1]

Communicated by Daniel W. Stroock, May 17, 1976

Certain models in quantum field theory can be defined by a generalized random process $\phi(f) = \int \phi(x) f(x)\, dx$ for $f \in S(R^d)$ satisfying the following conditions [3]: (a) Regularity. The expectation of $e^{\phi(f)}$ is entire analytic on $S(R^d)$; (b) Euclidean invariance (including reflections) of the underlying measure $d\mu$. This means that

$$\int \left[\prod_i \phi(f_i) \right] d\mu = \int \left[\prod_i \phi(\eta f_i) \right] d\mu.$$

Here $(\eta f)(x) = f(\eta^{-1}x)$ and η belongs to the Euclidean group. This identity induces a unitary transformation T_η on the space $E = L_2(d\mu)$ of random variables. (c) Reflection positivity. Let r denote reflection in the x_0 plane, and let v be a function of the random variables $\{\phi(f)\}$ where suppt f lies in the half space $x_0 > 0$. Then

$$\int \overline{v}(T_r v)\, d\mu \geqslant 0. \tag{2}$$

This final condition enables us to define the Hilbert space H (which plays the role of L_2 of the state space) and a contraction semigroup e^{-tH} (which defines the transition probabilities for the process). The inner product on H is given by (2) after dividing out by the space of null vectors. The semigroup e^{-tH} arises from translation in the $x_0 = t$ direction.

The simplest example of a process satisfying the above conditions is the Gaussian process whose generating functional is

$$\int e^{i\varphi(f)}\, d\mu_0 = \exp(-\tfrac{1}{2}\langle f, (-\Delta + 1)^{-1} f\rangle) L^2. \tag{3}$$

This process is known as the Ornstein-Uhlenbeck process. For $d = 1, 2$ we consider the following limiting process:

$$\int e^{i\phi(f)}\, d\mu^\pm = \lim_{\Lambda \uparrow R^2} \frac{\int e^{i\phi(f)} \exp(-P_\Lambda) \exp(-Q^\pm_{R^2\setminus\Lambda})\, d\mu_0}{\int \exp(-P_\Lambda) \exp(-Q^\pm_{R^2\setminus\Lambda})\, d\mu_0}. \tag{4}$$

AMS (MOS) subject classifications (1970). Primary 81A18.

[1]Supported in part by the National Science Foundation under Grant MPS74-13252.
[2]Supported in part by the National Science Foundation under Grant MPS75-21212.

where

$$P_\Lambda = \int_\Lambda [\lambda\colon \phi^4(x)\colon - \colon\phi^2(x)\colon]\, d^{(2)}x$$

and

$$Q_X^{\pm} = \int_X [\colon\phi^2(x)\colon \mp (4\lambda)^{-1/2}\phi(x)]\, d^{(2)}x.$$

When $d = 1$, $H = L^2(dx)$ and the generator H equals $-\frac{1}{2}d^2/dx - \frac{1}{2}x^2 + \lambda x^4$. In this case, the polynomial $Q^{\pm}$ has no effect on H and $d\mu^+ = d\mu^-$.

We study the case $d = 2$ and show that for small λ the limit exists and depends on $Q^{\pm}$. If we set $Q = 0$ and replace Δ in (3) by the Laplacian with Dirichlet boundary conditions on $\partial\Lambda$, we show that the translation group T_t defined on E does not act ergodically [1], hence the ground state of the corresponding Hamiltonian H is not unique. The effect of $Q^{\pm}$ is to choose a unique ground state. The dependence on boundary conditions is a phenomenon called phase transition in statistical mechanics and occurs, for example, in the Ising model. See [4].

The following results are established in [2].

THEOREM 1. *For small λ and for the quadratic boundary conditions $Q_{\pm}$ of* [2], *the limit $d\mu^{\pm}$ in* (4) *exists and defines a measure on $S(R^2)$ which is ergodic under R^2-translation with an exponential mixing rate $m > 0$. To define the exponential mixing rate m, let A and B be functions of $\phi(f)$ for $f \in C_0^\infty(R^2)$. Then*

$$m = \inf_{A,B} \lim_{|x|\to\infty} \left\{ \frac{\ln(\langle AT_xB\rangle - \langle A\rangle\langle B\rangle)}{|x|} \right\}.$$

Ergodicity combined with conditions (a)–(c) above are a mild strengthening of the Osterwalder-Schrader axioms. Verification of (a)–(c) gives our next result.

THEOREM 2. *The measure $d\mu^{\pm}$ satisfies all Osterwalder-Schrader axioms and has nonzero expectation value*

$$\int \phi(x) d\mu^{\pm} = \pm(4\lambda)^{-1/2} + O(\lambda^{3/2})$$

for $\lambda \ll 1$.

The above theorem is a nonuniqueness theorem because it implies $d\mu^+ \neq d\mu^-$. It also proves symmetry breaking, by showing that the symmetry $\phi \longleftrightarrow -\phi$ of the interaction is not a symmetry of the solution $d\mu^{\pm}$. The corresponding real time fields satisfy all Wightman axioms.

THEOREM 3. *The measure $d\mu^{\pm}$ has moments in the translated variable $\psi^{\pm} = \varphi \pm (4\lambda)^{-1/2}$ which are C^∞ as functions of $\lambda^{1/2}$, at $\lambda = 0$, and the derivatives $\partial/\partial(\lambda^{1/2})$ in the Taylor's expansion about $\lambda = 0$ can be evaluated explicity in terms* *of Feynman diagrams. The moments are also analytic in a small sector* $|\text{Im}\ \lambda| \leqslant \epsilon|\text{Re}\ \lambda| \ll 1$, $\epsilon \ll 1$.

REMARK. Theorem 3 makes precise the sense in which $d\mu^{\pm}$ is almost Gaussian.

BIBLIOGRAPHY

1. J. Glimm, A. Jaffe and T. Spencer, *Phase transitions for* ϕ_2^4 *quantum field*, Comm. Math. Phys. 45 (1975), 203–216.

2. ———, *A convergent expansion about mean field theory*, Ann. Physics (to appear).

3. K. Osterwalder and R. Schrader, *Axioms for Euclidean Green's functions.* II, Comm. Math. Phys. 42 (1975), 281–305.

4. D. Ruelle, *Statistical mechanics*, Benjamin, New York, 1969. MR 44 #6279.

DEPARTMENT OF MATHEMATICS, ROCKEFELLER UNIVERSITY, NEW YORK, NEW YORK 10021 (Current address of J. Glimm and T. Spencer).

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138 (Current address of A. Jaffe)