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Vorlesungen über Minimalflächen, by J. C. C. Nitsche, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 199, Springer-Verlag, Berlin Heidelberg, New York, 1975, xiii + 775 pp., $84.30.

It must have been in 1958 that I first heard of Nitsche's plan to write a book on minimal surfaces. It seemed an excellent idea, since the only two references describing 20th century contributions were Radó's 1933 monograph, *On the problem of Plateau*, and Courant's book, *Dirichlet's principle, conformal mapping, and minimal surfaces*, published in 1950. Both of these focused almost entirely on the solution of Plateau's problem. Furthermore, there had been a burst of activity in the late forties and throughout the fifties that led in a number of new and intriguing directions: Chen's study of branch points, Lewy's work on boundary regularity and analytic continuation of minimal surfaces, Bers' theorem on removable singularities and his theory of abelian minimal surfaces, Heinz's inequality on Gauss curvature, Finn's gradient estimates, Pinl's discussion of the Ricci condition in higher codimension, Shiffman's results on doubly-connected surfaces, Kruskal's bridge theorem, Fleming's example of a least-area problem with no finite-genus solution, Nitsche's elementary proof of Bernstein's theorem and his cylindrical version of it, Reid's isoperimetric inequality, and the reviewer's proof of Nirenberg's conjecture on complete minimal surfaces. "All this, and much, much more" you will find in Nitsche's book.

Right there we have two of the book's most striking characteristics. First, its scope, which is virtually encyclopedic, and second, its organization, allowing one to find quickly all references to any topic of interest. For example, if you wish to know more about any of the topics listed above, look under the author's name in the bibliography. Every item listed has individual page references to each place that it is cited in the text, a simple device, but extraordinarily useful in determining the content (or at least the context) of any of the 1232 papers and books listed in the bibliography. Since its usefulness far outweighs the effort needed in inserting these page references, I would hope that this practice might be widely adopted in future monographs.

Let me digress a moment to wonder about the mentality of an author who takes months or years to write a book, and then begrudges the few days needed to compile a careful index that will enhance immensely the book's value to its readers. Nitsche not only makes every effort to direct an interested reader to the relevant parts of his book, but he does equal service in the opposite direction; when citing a result from another book or paper, he takes unusual pains to give precise page numbers–one more courtesy to his readers that this reviewer, for one, truly appreciates.

But back to the main story. Having begun work on the book, Nitsche soon discovered that the subject was expanding more rapidly than he ever imagined. By the time each chapter was written, new results would appear that made it out of date, if not obsolete. This continued throughout the sixties, a decade which began with a whole new approach to minimal surfaces–that of geometric measure theory–and ended with the last steps in the classical solution of the Plateau problem: the proof of interior and boundary regularity of the solutions. When the manuscript was finally sent to the printer in 1972 it comprised what was to be 700 pages of text and 56 pages of bibliography. By the time it was set in print, another two years had passed. In a highly successful last-ditch attempt to bring the book up to date, Nitsche added a ten-page Appendix, describing briefly all the further relevant results that had come to his attention, either in the form of publications or preprints. Thus, as the book finally appeared in 1975, it represents a remarkably complete description of the state of the subject at that time, including most of what had been accomplished in the quarter century since the completion of Courant's book.

Before describing the subject matter in more detail, it is probably a good idea to note some of the topics *not* covered in the book. Most important is the author's decision to restrict himself almost entirely to surfaces in Euclidean 3-space. Thus, minimal surfaces of higher dimension or codimension are not studied, nor are minimal surfaces in Riemannian manifolds. The best reference for these subjects is *Lectures on minimal submanifolds*, Vol. I, by H. Blaine Lawson, put out in paperback by the Instituto de Mathematica Pura e Aplicada in Rio de Janeiro. This beautiful book is unfortunately not easy to obtain in most places.[1] The second major omission is the geometric-measure-theory approach to the subject. Again the best introduction is provided by a set of lecture notes by Lawson, *Minimal varieties in real and complex geometry*, published by the Presses de L'Université de Montréal in 1974. Of course, for those seeking the full treatment, there is also the book of Federer.

Perhaps the most useful way to start summarizing the contents of Nitsche's book is to compare it with other books on minimal surfaces, which are surprisingly few in number. In addition to those of Radó, Courant, and Lawson mentioned above, there are also *Plateau's problem: An invitation to varifold geometry* by Almgren, from 1966, and the reviewer's book, *A survey of minimal surfaces*, published in 1969. The latter covers selected topics from the research of the previous twenty years, with emphasis on the case of two-dimensional surfaces in Euclidean space of arbitrary dimension. Almgren's book provides an elementary introduction to some of the notions of geometric measure theory and their use in solving Plateau's problem. He does not include proofs, but tries to convey an intuitive understanding of this approach.

Nitsche's book includes two features that distinguish it from all the others. The first is an extensive historical account of the whole subject, as well as of each of the topics treated. The second is the inclusion of a substantial number of specific examples of minimal surfaces and classes of surfaces that can be

[1] I understand, however, that it is to be reissued by Publish or Perish Press, and will become generally available.

represented by explicit formulas. There are five sections entitled *Special minimal surfaces* scattered through the book in which these examples are given together with detailed descriptions of their properties. It is, in fact, one of the beauties of the subject, that it abounds in concrete examples of surfaces exhibiting interesting properties of their own, and furthermore serving as illustrations of (as well as inspirations for) general theorems.

As with most of the previous books on minimal surfaces, the Plateau problem occupies a central position. The longest chapter of the book by far, Chapter V (191 pages), is devoted almost entirely to Plateau's problem in its classical formulation: "find a parametric minimal surface of the type of the disk bounded by a given Jordan curve." One might well ask why such attention should be given to a problem purportedly resolved in 1930 in the work of Douglas and Radó. There are, in fact, several good reasons: the problem was *not* completely resolved at the time; many related questions remained unanswered; and interesting new methods have been applied to the problem.

For many years it came as a great surprise to each newcomer to the subject to learn that the classical solution had a serious gap: neither Douglas nor Radó had been able to show that the mappings obtained by their methods were free of singularities. They were forced to enlarge the class of admissible surfaces to include the existence of possible "branch points", in order to obtain a solution. The question of whether such branch points occur on the solution surfaces remained open for forty years.

Another unsolved problem was that of boundary regularity. The mappings defining the solution surfaces were shown to be continuous in the closed disk, but one would expect them to possess some degree of differentiability if the given boundary curve was sufficiently differentiable.

It was not until the late sixties that the first significant progress was made on these problems of interior and boundary regularity, by the reviewer and S. Hildebrandt, respectively. Then in a relatively short time, the entire question was settled most satisfactorily by the combined work of a number of authors, so that one now has the following definitive result: let γ be a Jordan curve of class $C^{m,\alpha}$ in $\mathbf{R}^3$, where $m \geqslant 1$, $0 < \alpha < 1$. Let Δ be the closed unit disk, D its interior, and C its boundary. Then there exists a map $f\colon \Delta \to \mathbf{R}^3$ with the following properties:

(i) f is of class $C^{m,\alpha}$ on Δ,

(ii) f restricted to C is a homeomorphism onto γ,

(iii) f restricted to D is a real analytic immersion onto a minimal surface S,

(iv) f restricted to D is a conformal map onto S,

(v) S has least area among all surfaces defined by maps of Δ into $\mathbf{R}^3$ satisfying (ii).

Nitsche's timing was fortunate with respect to boundary regularity (property (i) above) for which he is able to include a complete proof, but the solution to interior regularity came too late, and the reader has to settle for references to the original literature given in the Appendix.

Does a definitive solution such as this mean the subject is thereby closed?

Definitively not! Among the many questions remaining, we may note the following.

1. Under what conditions on γ will the solution be unique?

2. For cases of nonuniqueness, can one describe the totality of solution surfaces? Can one give a bound on the number of distinct ones? Can there exist a continuum of solution surfaces?

3. The solution surfaces are immersions, but may have self-intersections. Under what conditions on γ can one find a solution surface that is free of self-intersections?

4. If the curve γ has a "corner" formed by two arcs meeting at a well-defined angle, can one describe the behavior of the solution surface at that corner? For example, does the tangent plane to the surface tend to the plane formed by the tangents to the two arcs at the corner?

5. For surfaces of minimum area, can there be boundary branch points?

6. Can one give constructive existence proofs for the solutions of Plateau's problem?

7. To what extent does one have continuous dependence of the solution surface on the curve γ?

Various of these results would have still further interesting consequences. They are all currently under active investigation.

The question of uniqueness, like that of regularity, went for forty years with essentially no progress. There were some early results of Radó, and nothing further until the recent result of Nitsche that if γ is a regular analytic curve whose total curvature does not exceed 4π, then the solution is unique. Substantial results on question 2 were obtained only recently by Böhme and Tomi, including the fact that a regular analytic curve can bound only a finite number of surfaces of minimum area. Also Tromba, in work not yet published, has devised an entirely new approach to this question.

Progress on question 3 has been made recently by Gulliver and Spruck. Some results on 5 are due to Gulliver and Lesley, while Beeson has worked on 4 and 6, and both Beeson and Böhme have studied 7. All of this work was too late to be included by Nitsche in his main text, although some appeared in time to be cited in the Appendix.

Besides this most classical of Plateau problems, Nitsche treats a number of others: unstable solutions, free boundary-value problems, "movable" boundary-value problems (involving a fixed piece of twisted wire whose ends are joined by a length of thread), doubly-connected surfaces bounded by a pair of prescribed Jordan curves, and other variants. The number of different techniques that have been applied to these problems is astonishing. Besides the classical methods of calculus of variations, differential geometry, and complex variables, there are more recent ones of Morse theory, Sobolev spaces, and Banach manifolds. It is certainly one of the fascinations of working with minimal surfaces to see each new method applied to obtain new results.

Although Plateau's problem is the largest single topic, it is by no means the only one given serious attention. Another major component of minimal surface theory, represented by another long chapter in the book, is the approach via partial differential equations. A function of two variables whose graph is a minimal surface satisfies a nonlinear elliptic differential equation; much fascinating work was done in the sixties, revealing an interplay between the ellipticity, which leads to many familiar properties of solutions, and the nonlinearity, which results in some unexpected quirks of behavior. The principal contributors to this were Finn, Jenkins and Serrin, with further important results by Nitsche, de Giorgi and Stampacchia.

One more development of the sixties was a theory of complete minimal surfaces: relations between their topological and conformal structure, and properties such as total curvature and image under the Gauss map. This too is the subject of a chapter in Nitsche's book.

The main text of the book may be said to end on p. 632. There follows first a beautiful chapter containing a miscellaneous collection of theorems, remarks, anecdotes, references, and a list of 95 open problems; second, the Appendix referred to several times before, reporting on the most recent results that appeared too late to be included in the text; and finally, the 56-page indexed bibliography.

The work in its entirety bespeaks a monumental effort. I was repeatedly astonished by the thoroughness with which the author tracked down references and checked out different treatments of a topic. One might fear that such an approach would at the same time be heavy-handed, but that is not at all the case. The style is generally informal, and makes for enjoyable reading. The fact that it is written in German will undoubtedly be a problem for many potential readers. That is one of the accidental side effects of the time span over which the book was written, since it was started not long after the author arrived in this country, when he felt more at ease writing in his native language.

What more can one say? Obviously, in any book of this size, there are bound to be places where one might prefer an alternative treatment, a different emphasis, or a modified historical account. However, these would be quibbles compared to the whole.

Happily, the subject goes right on developing. One of the most promising new approaches seems to be the application of the methods of global analysis to the solution of Plateau's problem, as in current work of Böhme, Tomi, Tromba, and Uhlenbeck. Whatever new results may come out of this and other work, it seems clear that Nitsche's book will be an invaluable aid to anyone working in the field for years, and probably decades to come.

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