details of the interaction is called *universality* and has important consequences both for physics and mathematics, as we now explain.

A typical correlation length in statistical mechanics might be 10^3 times the atomic spacing. This means that on the distance scale of atomic spacing, statistical mechanics is typically near its critical point, hence independent of many of the details of the intermolecular forces, hence governed by the "general" laws of physics (as opposed to "compound dependent" laws of chemistry).

The significance of universality to mathematics is that it indicates the existence of a general theory, whose qualitative (and quantitative) features describe a broad range of phenomena. This theory, once completed, might belong to the subject of non-Gaussian stochastic processes with index space R^d or Z^d, $d \geqslant 2$. For example the $d = 2$ Ising model critical point, seems to be related to a theory of random nonoverlapping closed curves in the plane, and thus to a two dimensional generalization of the Poisson process.

Thompson's book is elementary, both in its mathematical and its physical content. The reviewer found that it served well as a text for portions of an introductory mathematical physics course. It is also a good companion to the mathematically more advanced book by Ruelle [**1**] in providing some of the motivation and insight which are valuable to mathematicians working on this interdisciplinary field. Chapter 4 is an introduction to phase transitions and critical phenomena in terms of simple solvable models such as the van der Waals gas and the mean field magnet. Chapters 5 and 6 are the core of the book. They present a pleasant account of the exactly solvable two dimensional Ising model as an illustration of phase transitions and critical phenomena, following the method of [**2**]. Chapter 7 contains an application of the Ising model to the role of hemoglobin in the transport of oxygen.

The Lenard book is at the level of a graduate seminar. There is an excellent introductory article by Lanford in this volume which does not require prior knowledge of the subject and should be accessible to a graduate student with a background in probability and/or functional analysis. The series by Domb and Green is also a collection of individual articles. These articles are at the level of advanced monographs, but again the opening article by Griffiths provides a good general introduction to the subject.

References

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Applications of algebraic topology, by Solomon Lefschetz, Applied Mathematical Sciences, vol. 16, Springer-Verlag, New York, Heidelberg, Berlin, 1975, viii + 189 pp., $9.50.

Are there applications of algebraic topology? Certainly a subject, conceived by Riemann and delivered into the world by Poincaré, ought to have applications. But in mathematics, as in life, the child is often quite unlike the parents. Are there applications of *modern* algebraic topology? The mere mention of such a question will elicit, no doubt, strong protest from those who would argue that there is no need to justify mathematics. Such arguments are based on philosophy; we are intent on practicality. The fact is that there has been, in recent years, a concerted effort to associate the word "applied" to divers areas of mathematics. Moreover (and this is the crucial point), the general public, and to some extent the mathematical public, construes these efforts as an apology for mathematics. Whether or not we should ask the question is moot. It has been asked and answered. What deserves our attention is the answer.

A glance at the title shows that Lefschetz believed there were indeed applications of algebraic topology. However, we ought to emphasize that this book is definitely not a philosophical treatise, espousing the virtues of applicable mathematics. It is merely an exposition of two applications of algebraic topology, written by Lefschetz in his "unique and vigorous style". (Those familiar with this style will immediately recognize the euphemism.) Yet, Lefschetz *did* believe there were applications of algebraic topology, and, because he was undeniably a master of his subject, his belief will be interpreted as a definitive answer.

It is an answer which needs some clarification. The truth is that any area of mathematics has applications if the word "applications" is sufficiently vague. What do we mean by applications? This seems to be a difficult question for applied mathematicians themselves, since no matter what definition is used there always remain some areas of applied mathematics which are not applicable. They prefer to be vague. But if we are honest with the public and ourselves we will use our words precisely. Since most people equate the concepts of applicability and utility, we shall do so here. An application of algebraic topology ought to be useful–at least to somebody!

Now, having accepted the verdict, we can examine the evidence. What are the applications of algebraic topology? Lefschetz presents two, which we briefly describe.

Classically, an electrical network is simply a graph, consisting of nodes and (directed) branches joining the nodes, together with two functions which assign currents and voltages to each of the branches. Of course, the current and voltage distributions must satisfy Kirchoff's two laws: that the algebraic sum of the currents at each node is zero, and that the algebraic sum of the voltages around any loop is zero. From this quite simple idea a very pretty theory of networks develops. Lefschetz points out that this theory is merely the "first chapter of classical algebraic topology".

To see this, we need only translate the notions of current and voltage distribution into the language of algebraic topology. The graph of a network we immediately recognize as a simplicial complex (of dimension 1). If we label the branches of the graph as $\{b_j\}$ then we can think of a current distribution as a formal linear combination $\sum i_j b_j$, where i_j is the current assigned to b_j. The collection of all such formal linear combinations (with real coefficients) is a vector space, and we recognize this as C_1, the 1-chains of the simplicial complex. But which 1-chains satisfy Kirchoff's first law? To see this, we consider the 0-chains C_0, the vector space of all formal linear combinations of the nodes. There is a linear map $\partial: C_1 \to C_0$ which takes a branch to the difference of its endpoints. Now it is easy to verify that a 1-chain satisfies Kirchoff's first law precisely when $\partial(\sum i_j b_j) = 0$. In the language of algebraic topology, a current distribution is a 1-cycle. We can similarly interpret a voltage distribution by considering the dual vector spaces together with the dual map $\delta: C_0^* \to C_1^*$. We view a voltage distribution as a formal linear combination $\sum v_j b_j^*$, where b_j^* is dual to b_j and v_j is the voltage assigned to b_j. This time we can argue that $\sum v_j b_j^*$ satisfies Kirchoff's second law precisely when it is in the image of δ. A voltage distribution is, therefore, a 1-coboundary. The translation, from the ordinary language of networks to algebraic topology, is complete.

Is this an application of algebraic topology? Is it useful? In fact, Lefshetz *does* use the new language to prove one result, on duality of networks, but the result, while mathematically aesthetic, is of doubtful use to the electrical engineer. (Perhaps a more serious objection is that the proof can be given without the mention of algebraic topology.) Moreover, useful or not, that which is applied here is not algebraic topology, but rather the *language* of algebraic topology; and the most elementary language at that.

One certainly cannot apply such criticism to the second application which Lefschetz discusses, to the theory of Feynman integrals. These are integrals which arise as coefficients of a perturbation series in quantum field theory, and were used by Feynman to describe the elements of the scattering matrix. Setting the physical origin aside, we are presented with a very beautiful (and difficult) mathematical problem. A Feynman integral is, in general, a multiple integral, over real contours, of an integrand which depends on certain complex parameters. For fixed values of the parameters the integrand, which is, in general, a rational function, will have certain singularities, and naturally the real contour must avoid these. The integral then defines a function (multi-valued) of the parameters. In the modern theory the problem is to find the singularities of this function and to determine its analytic nature near these singularities.

We can illustrate this problem with a very simple example. We define a function by

$$F(t) = \int_\Gamma \frac{dz}{z^2 - t},$$

where t is a complex parameter and Γ is a closed curve in the z-plane. For a fixed value of t the integrand has singularities at $z = \pm\sqrt{t}$, and clearly the value of $F(t)$ depends on Γ, in the usual way, but does not change if we deform Γ, avoiding the singularities. Then how can $F(t)$ be singular? Suppose we start at some positive real t and some curve Γ, and let t approach zero, deforming Γ so as to avoid the singularities. Now as t approaches zero the two singularities at $z = \pm\sqrt{t}$ approach one another and coalesce. If our original curve Γ does not "contain" either singularity, then this makes no difference, and $F(t)$ is not singular at the origin. But, if the original curve *does* "contain" one of the singularities then as the singularities coalesce, Γ will be "pinched" between them and thus $F(t)$ is singular at the origin. In other words, whether or not $F(t)$ has a singularity at $t = 0$ depends on which "sheet" of $F(t)$ we consider. What is the nature of $F(t)$ near a singularity? Again consider a positive, real t and let Γ be a small circle once around $+\sqrt{t}$ in a clockwise direction. Now if t circles once around the origin in the t-plane then the singularities $z = \pm\sqrt{t}$ circle half-way around in the z-plane; that is, they are interchanged. Our original Γ around $+\sqrt{t}$ will become a small circle, once around $-\sqrt{t}$, and it is easy to verify that $F(t)$ changes sign.

The general problem is similar. What are the singularities? What is the nature of the function near the singularities? Of course, in higher dimensions there are many difficulties which are not apparent in our simple example. In particular, the integrand is now singular on varieties, rather than points, and the contours are higher dimensional "cycles", rather than curves. By analogy we must determine when these cycles are "pinched" by coalescing singularities, and how they are affected when the parameters "circle around" a singularity. To an algebraic topologist it is not surprising that the answers to these questions can be described using homology and cohomology, together with the intersection pairing. In fact, the answer is given by a theorem of Picard and Lefschetz. which anticipated the work of Feynman by many years. The theory, begun by Picard and developed by Lefschetz, has been extended and applied in recent years by Fotiadi, Froissart, Lascoux and Pham. It is a very pretty subject indeed.

There have also been some surprises, and I would be remiss in not mentioning one in particular. An immediate generalization of our simple example above is the integral over a real contour of the function

$$(z_1^{a_1} + z_2^{a_2} + \cdots + z_n^{a_n} - t)^{-1},$$

where the exponents are positive integers. As before, we are interested in the analytic nature of the function near $t = 0$, and to this end, we are interested in the variety

$$V(a_1, a_2, \ldots, a_n) = \{(z_1, z_2, \ldots, z_n) | z_1^{a_1} + z_2^{a_2} + \cdots + z_n^{a_n} = 0\}$$

near the origin. Now the intersection of such a variety with a small ball B about the origin is just the cone on the intersection of the variety with the sphere bounding the ball. The fact is that the closed manifold $V(a_1, a_2, \ldots, a_n) \cap \partial B$ is often homeomorphic to a sphere, but not diffeomorphic to the usual sphere. In short, we have examples of "exotic spheres!" It seems remarkable that this chain of ideas, as spurious as it is, from particle physics to exotic differentiable structures on spheres, can be outlined in one short paragraph.

We have now come to a rather troublesome question: Is *this* an application of algebraic topology? Are Feynman integrals useful? Perhaps the pertinent question ought to be: Useful to whom? That Feynman integrals, or at least the calculations associated to them, are useful to algebraic topologists is certainly clear (viz., examples of exotic spheres). That Feynman integrals are useful to mathematical physicists is somewhat less clear, since the mathematical techniques used to describe their analytic nature seem to have advanced beyond the physicist's ability to interpret these results. This may be a "mathematization of physics" that many physicists would rather do without. And whether or not they are of use to mathematical physicists, this is not what *most* people mean by useful. It would be incorrect to say that Feynman integrals are useless, but it would be misleading to say that they are useful.

The same can be said for algebraic topology, and if you would argue the point, then argue it with a well-educated layman–choose a businessman or dean, or perhaps any taxpayer–and be prepared to answer these questions. What are the tangible results? If algebraic topology had never been developed would there be one less transistor or one more sickness? Would physics or chemistry or biology be any different? The answers are obvious. The simple truth is that if we use our words with any precision, then algebraic topology has no applications. We should not be proud of such a fact; nor should we be ashamed. Moreover, it is entirely possible that some day, through the proverbial long chain of reasoning, algebraic topology will prove to be useful. But if we make the claim that algebraic topology, as presently constituted, has utility, then we must expect to eventually incur the wrath and indignation of those we have duped. Algebraic topology is simply not useful!

Then what is the intent of this book by Lefschetz? Should we search for applications no matter how feeble the results? There is a superficial appeal to such an effort, for mathematics which claims utility as its aim is undeniably popular. (Witness the recent public reaction to Thom's *Castastrophe theory*, although in this case popularity was certainly not the motivation.) Yet the conscious search for applications of an area of mathematics has historically met with failure. Mathematics seems to yield applications at her own whim. To search for applications of a developed subject is not only futile, but also will likely produce poor applications and even poorer mathematics.

"To do mathematics you need a problem". Felix Klein meant more than the problem of finding a problem. Mathematicians in general, and algebraic topologists in particular, should concentrate on *doing* mathematics, and content themselves with *recognizing* the applications, even if they are merely "applications" to mathematics itself–indeed, *especially* when they are to mathematics itself. Now herein lies the difficulty. All too often the modern mathematician tends to be an intellectual bigot, provincial in outlook and learning. Perhaps we can no longer be universalists, but neither should we be pedants. If we hope to *recognize* the applications, or even hope to appreciate the beauty and unity of mathematics, we must broaden our perspectives and hone our intellectual curiosity. Perhaps algebraic topologists should learn some mathematical physics, and mathematical physicists some algebraic topology; not because it is useful, but because it is interesting! It seems that Lefschetz agreed.

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Geometry of Banach spaces–Selected topics, by Joseph Diestel, Lecture Notes in Mathematics, no. 485, Springer-Verlag, Berlin, Heidelberg, New York, 1975, xi + 282 pp., $11.50.

Beginning with Banach's *Operations lineaires* the study of Banach spaces has been a pursuit of classification. This can be in terms of the classically important spaces as $C(K)$ or $L_p(\mu)$, or it can be in terms of certain desirable internal conditions on norm and specified elements such as smoothness and convexity properties, or it can be in terms of external conditions, for example, on dual spaces, subspaces, or factorization of operators. An elegant example of the first is the Bohnenblust-Kakutani result that a Banach lattice is linearly isometric to $L_p(\mu)$ or a sublattice of $C(K)$ if and only if $\|x + y\|^p = \|x\|^p + \|y\|^p$ or $\|x + y\| = \max(\|x\|, \|y\|)$ whenever $x \wedge y = 0$. Smoothness refers to the existence of unique supporting hyperplanes to points on the surface of the unit ball and is usually phrased in terms of the differentiability of the norm. Uniform convexity describes the shape of the surface of the unit ball. These concepts are important to minimization problems in optimization theory and P.D.E. among others. These conditions are "geometric" in nature and are not, in general, preserved under isomorphisms. However, deep studies have been made into various Banach space properties which imply smoothness or convexity conditions under some equivalent norm. Perhaps the deepest and most important of these is the one obtained by P. Enflo which states that a Banach space X has an equivalent norm under which it is uniformly convex if and only if it is superreflexive (i.e., every Banach space Y which has the property that every finite dimensional subspace of Y is almost isometrically embeddable into X is itself reflexive). This is an elegant blending of the internal geometry (uniform convexity) with the external geometry (superreflexivity). There are many examples of the third type mentioned above. For example, the theory of $\mathcal{L}_p$ spaces (roughly, a Banach space is a $\mathcal{L}_p$ space if it is the union of an upwards directed family of finite dimensional spaces each uniformly equivalent to an $l_p(n)$, $1 \leqslant p \leqslant \infty$). Thus there is the elegant Lindenstrauss-Pełczyński-Rosenthal result that X is a $\mathcal{L}_p$ space or a Hilbert space if and only if it is isomorphic to a complemented subspace of an $L_p(\mu)$ space, $1 \leqslant p < \infty$. In duality there is the Lindenstrauss-Rosenthal result that X is a $\mathcal{L}_p$ space if and only if X^* is a $\mathcal{L}_{p'}$ space, $1 \leqslant p \leqslant \infty$. In factorization of operators, there is the beautiful theorem of Davis, Figil, Johnson and Pełczyński that a weakly compact operator factors through a reflexive space. One also has the deeply significant work of many authors (Lindenstrauss, Pełczyński, Nikišin, Stein, Rosenthal, Maurey, and others) on absolutely p-