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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

Mathematical statistical mechanics, by Colin J. Thompson, The Macmillian Company, New York, 1972, ix + 278 pp., $15.95.

Statistical mechanics and mathematical problems, *Battelle*, *Seattle*, 1971 Rencontres, Edited by A. Lenard, Springer-Verlag, Berlin, Heidelberg, New York, 1973, v + 246 pp., $9.90.

Phase transitions and critical phenomena, Edited by C. Domb and M. A. Green, Academic Press, London, New York, Vol. 1, *Exact results*, 1972, xv + 506 pp., $31.00. Vol. 2, 1972, xv + 518 pp., $31.00. Vol. 3, *Series expansion for lattice models*, 1974, xviii + 694 pp., $46.50.

Among the major scientific issues having a mathematical component, there are a number for which the process of quantification and model building is incomplete (e.g. economics, genetics and ecology). In contrast are issues which can be clearly formulated (but not yet solved) in mathematical terms. Mathematical physics has a particularly rich collection of this second class of problems; we mention the instabilities of plasmas, the singularities of space time allowed by Einstein's equations for general relativity, turbulence, the renormalization of quantum fields, and the theory of critical behavior in statistical mechanics. The importance of these problems to physics is clear. Their importance to mathematics lies in the expectation that their solution will require new developments in–or perhaps even new branches of–mathematics.

Statistical mechanics concerns infinite systems, and the relevant analysis is analysis over infinite dimensional spaces. The simplest measure over an infinite dimensional space is an infinite tensor product $d\mu = \otimes_{i=1}^{\infty} d\mu_i$ of measures $d\mu_i$ over finite dimensional spaces (e.g. R^d or $\{-1, +1\}$). These measures exist under the very general hypothesis

$$\sum_i \left|1 - \int d\mu_i\right| < \infty.$$

They are not very interesting because they correspond to situations in which there is no interaction between the particles or between the degrees of freedom of the problem. However the tensor product measures are not completely misleading, because the measures of statistical mechanics typically are almost tensor products.

The essential postulate necessary for statistical behavior is not statistical independence. It is statistical almost independence. The probability distribution $d\mu_i$ of the ith particle depends primarily on the positions of a finite number of its neighbors (short range stable forces, in physics terminology), and depends only very weakly or not at all on the remaining infinite number of particles. Thus the proper definition of $d\mu$ is not the tensor product definition $\lim_{n\to\infty} \otimes_{i=1}^{n} d\mu_i$, but rather $d\mu$ is a limit of measures $d\mu^{\Omega}$ defined over finite dimensional spaces (e.g. Ω^n or $\{-1, +1\}^n$). $d\mu^{\Omega}$ is determined by the restriction of n particles confined to a region Ω of space, of volume V, and the limit $V \to \infty$ is called the infinite volume, or thermodynamic limit. For many purposes, it is sufficient to study the free energy, defined as

$$f(\rho) = \lim_{V} \left(\ln \int d\mu^{\Omega}\right)/V$$

with the density $\rho = n/V$ held fixed. The "almost tensor product structure" of $d\mu^{\Omega}$ is the main hypothesis required for the existence of this limit.

There is an essential difference in the analysis over finite vs. infinite dimensional spaces. The same differences occur in random processes $\xi(t)$, $t \in R$ vs. random fields, $\xi(x)$, $x \in R^d$, $d \geqslant 2$ and in elliptic operators in a finite vs. infinite number of degrees of freedom. The difference is that in the infinite dimensional case, the solution may depend discontinuously on the parameters of the problem. The discontinuity is not an irritating pathology, to be removed by an appropriate reformulation. Rather, it is the central and dominant feature of the problem. Let $f^{\Omega}(\rho) = (\ln \int d\mu^{\Omega})/V$. Then $f^{\Omega}(\rho)$ turns out to be an analytic function of ρ (and of the temperature, and the other parameters contained in the definition of $d\mu^{\Omega}$). However $f(\rho)$ should be only piecewise analytic. The boundaries between the analytic pieces of f correspond to the phase transitions for example between gas and liquid or liquid and solid states, and thus are important to the structure of $d\mu$. The existence of these phase transitions has never been established in realistic models of fluids, but it has been established in simplified models of magnetism, such as the Ising model.

In magnetic models, the phase transition is associated with a lack of statistical independence between the ith and jth particles at infinite separation. In the simplest case $i \in Z^d$, $d\mu_i = (\delta_{-1} + \delta_{+1})/2$ (Dirac delta functions). Let $\mathfrak{N}(\Omega)$ be the set of nearest neighbor pairs (i, i') in Ω, i.e. pairs (i, i') for which $\operatorname{dist}(i, i') = 1$. Then we define

$$d\mu^{\Omega} = \prod_{i\in\Omega} e^{h\sigma_i} \prod_{(i,i')\in\mathfrak{N}(\Omega)} e^{-J\sigma_i\sigma_j} \prod_{i\in\Omega} d\mu(\sigma_i)$$

with h real and $-J > 0$. This case is the Ising model. We differentiate between weak coupling ($J \simeq 0$), and strong coupling ($-J \gg 1$), but because in both cases only nearest neighbor pairs (i, i') interact, $d\mu^{\Omega}$ has an almost tensor product structure and the existence of $d\mu$ follows.

The uniqueness question is much deeper than the existence question and is related to jump discontinuities in the derivative of f across the boundaries between its analytic pieces. A complete definition of $d\mu^{\Omega}$ requires a specification of the boundary spins

$$\sigma_i,\ i \in \partial\Omega \equiv \{j\colon \operatorname{dist}(j, \Omega) = 1\}.$$

If the coupling is weak (high temperature) then $d\mu$ is independent of boundary conditions, and is unique. If the coupling is strong (low temperature) and $d \geqslant 2$, d does depend on the boundary conditions, for $h = 0$. Furthermore $f(J, h)$ is real analytic in J and h for $h \neq 0$ and $\partial f/\partial h$ has a jump discontinuity at $h = 0$, so that a phase transition occurs for $h = 0$, $-J \gg 1$. The connection between phase transition and the dependence of $d\mu$ on boundary conditions is quite general, and relates to the theory of convex functions on infinite dimensional spaces.

Since $d\mu$ is defined by the limit $\Omega\uparrow R^d$, $\partial\Omega \to \infty$, we see that weak coupling leads to (exact) independence of $\partial\Omega$ and $d\mu$, i.e. independence of random variables separated by an infinite distance, while strong coupling allows phase transition and dependence between random variables separated by an infinite distance. The separation distance $|i - j|$ is the Euclidean distance between the labels $i, j \in Z^d$ for the degrees of freedom. Thus phase transitions and this sharp dichotomy between strong and weak coupling can exist only for infinite systems; in finite systems, there is no "infinitely distant separation" between the degrees of freedom–i.e. between the coordinate directions in the finite dimensional space of variables defining the problem.

Other mathematical approaches to phase transitions involve ergodic theory and spectral theory. In the ergodic theory approach, $d\mu$ is decomposed into ergodic components relative to the action of the lattice symmetry group. These irreducible components are called pure phases, and the possibility of more than one pure phase (depending on boundary conditions) corresponds to phase transitions. In the spectral theory approach, a distinguished lattice direction is selected, and the conditional expectation onto successive normal hyperplanes (unit separation) defines an operator T, called the transfer matrix. Degeneracy of the largest eigenvalue of T corresponds to nonergocitiy of $d\mu$ and hence to phase transitions. In all approaches one finds an interplay between functional analysis, complex analysis and combinatorial analysis.

The critical point concerns the transition from weak to strong coupling. More precisely, for the Ising model, the critical value $J = J_c$ of J is defined to be the supremum of the J's for which the free energy $f = f(J, h)$ has a jump discontinuity (phase transition) at $h = 0$. In a neighborhood of the critical point, $J = J_c$, $h = 0$, f cannot be analytic in either h or J, but the leading singularities dominate the behavior of f near the critical point. These singularities are expected to be independent of all but a few topological features of the interaction (dimension, spin, . . .). This independence of f from the details of the interaction is called *universality* and has important consequences both for physics and mathematics, as we now explain.

A typical correlation length in statistical mechanics might be 10^3 times the atomic spacing. This means that on the distance scale of atomic spacing, statistical mechanics is typically near its critical point, hence independent of many of the details of the intermolecular forces, hence governed by the "general" laws of physics (as opposed to "compound dependent" laws of chemistry).

The significance of universality to mathematics is that it indicates the existence of a general theory, whose qualitative (and quantitative) features describe a broad range of phenomena. This theory, once completed, might belong to the subject of non-Gaussian stochastic processes with index space R^d or Z^d, $d \geqslant 2$. For example the $d = 2$ Ising model critical point, seems to be related to a theory of random nonoverlapping closed curves in the plane, and thus to a two dimensional generalization of the Poisson process.

Thompson's book is elementary, both in its mathematical and its physical content. The reviewer found that it served well as a text for portions of an introductory mathematical physics course. It is also a good companion to the mathematically more advanced book by Ruelle [**1**] in providing some of the motivation and insight which are valuable to mathematicians working on this interdisciplinary field. Chapter 4 is an introduction to phase transitions and critical phenomena in terms of simple solvable models such as the van der Waals gas and the mean field magnet. Chapters 5 and 6 are the core of the book. They present a pleasant account of the exactly solvable two dimensional Ising model as an illustration of phase transitions and critical phenomena, following the method of [**2**]. Chapter 7 contains an application of the Ising model to the role of hemoglobin in the transport of oxygen.

The Lenard book is at the level of a graduate seminar. There is an excellent introductory article by Lanford in this volume which does not require prior knowledge of the subject and should be accessible to a graduate student with a background in probability and/or functional analysis. The series by Domb and Green is also a collection of individual articles. These articles are at the level of advanced monographs, but again the opening article by Griffiths provides a good general introduction to the subject.

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BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

Applications of algebraic topology, by Solomon Lefschetz, Applied Mathematical Sciences, vol. 16, Springer-Verlag, New York, Heidelberg, Berlin, 1975, viii + 189 pp., $9.50.

Are there applications of algebraic topology? Certainly a subject, conceived by Riemann and delivered into the world by Poincaré, ought to have