There are a number of typographical errors in Shafarevich's book. In particular, many of the bibliographical references are misnumbered. There is one mathematical error: in the definition of a scheme (p. 244) one must consider *locally* ringed spaces, and require that all morphisms induce *local* homomorphisms of the stalks.

In conclusion, we can say that Dieudonné's history should be read by everyone interested in algebraic geometry, and that Shafarevich's book—at least until the publication of some other introductory algebraic geometry texts now in preparation—is a serious contender for "the best modern introduction to algebraic geometry".

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Foundations of special relativity: *Kinematic axioms for Minkowski space-time*, by J. W. Schutz, Lecture Notes in Mathematics, No. 361, Springer-Verlag, Berlin, Heidelberg, New York, 1973, xx+314 pp.

Since the latter part of the nineteenth century, the papers on the axiomatic foundations of Euclidean geometry and the closely related projective, affine, hyperbolic, and elliptic geometries are to be numbered in the thousands. The absence of a comparable flow of papers about the axiomatic foundations of special relativity is hard to understand, especially in view of the fact that Einstein's basic theoretical paper on special relativity appeared in 1905 [**2**], which is close to the heyday of foundational studies of Euclidean geometry.

During the early part of this century, almost the only person doing any work on the qualitative foundations of special relativity was Alfred A. Robb, who first began publishing on the subject in 1911, followed by a small book in 1913, a revision of that book in 1921, and a full-scale work in 1936 [**5**]. Robb's axiomatization of the geometry of special relativity is important for several reasons. First of all, he uses an extremely simple single primitive concept, the binary relation of one space-time event's being *after* another. This is a simpler primitive in logical structure than any of those that have been used for the foundations of Euclidean geometry, and for good reason. Tarski showed many years ago that no nontrivial binary relation can be defined in Euclidean geometry and consequently there is no hope of basing Euclidean axioms on a binary relation between points.

On the other hand, the complexity of Robb's axioms stands in marked contrast to the simplicity of the single primitive concept. If I cited the full set of axioms here, the reader would be appalled by their length and, in many cases, relative difficulty of intuitive comprehension.

Shortly after World War II, A. G. Walker in several publications [**9**], [**10**] offered a new qualitative foundation of the geometry of special relativity. In addition to the set of space-time events, he used particles, an ordering relation of beforeness on events, and, perhaps most importantly, a one-one signal-mapping from one particle onto another. Perhaps the most unsatisfactory feature of Walker's approach is that the signal-mappings are complex functions that do at one stroke work that should be done by a painstaking buildup of more elementary and more intuitively characterizable operations or relations. As part of his approach, several of the axioms are very powerful, and one has the feeling that weaker and more qualitative axioms could be found within his framework.

The book under review is an important and useful contribution to the small literature on the axiomatic foundations of special relativity. Schutz builds very much on the work of Walker, using *particles*, which correspond physically to inertial particles, and the *signal relation*, which corresponds physically to signaling by means of light rays.

Schutz's axiom system goes beyond that of Walker's in two respects. First, Walker did not state strong enough axioms to characterize completely Minkowski space-time, and secondly, more than half the axioms have a form that is not closely related to any of those given by Walker and which have, in several cases, a rather strong appeal in terms of physical intuition. As Schutz indicates, the most important immediate predecessor of his book is the article by Szekeres [**6**], whose approach resembles that of Walker, but Szekeres treats both particles and light signals as objects rather than treating light signals as formally being characterized by a binary relation of signaling. A very similar, independent treatment based upon the binary symmetric relation of signaling appeared almost simultaneously in Latzer [**4**], although Latzer did not work out as complete an axiomatization as that given by Schutz.

Schutz's formulation is based upon eleven axioms, the first five of which are similar to axioms already given by Walker. The final six axioms are more powerful and primarily depend upon a concept that is geometrically very natural but that has not ordinarily been adequately exploited in the axiomatizations of special relativity. This is the concept which Schutz calls a *spray* of the set of inertial paths that pass through a given point, in other words, the set of inertial paths that are contained in the light cone whose vertex is the given point. Schutz refers to a spray as the set of inertial particles whose paths have the properties mentioned. Obviously in both his and Walker's development the concept of an inertial particle is not really a physical but a geometrical concept, and one could just as well speak of inertial paths as of inertial particles. The four essential properties of sprays expressed in the axioms are the following: (i) between any two distinct particles of a spray, there is a particle which is distinct from both; (ii) each spray is isotropic; (iii) there is a spray with a maximal symmetric subspray of four distinct particles (this is the dimension axiom); and (iv) each bounded infinite subspray is compact.

The remaining two axioms (of the six) postulate that space-time can be "connected" by particles and that given any two distinct particles coinciding at some event, that is, intersecting at some space-time point, there is another particle which forms a third side of the "triangle."

As this informal discussion should make clear, I believe that each of Schutz's axioms has a clear intuitive content. On the other hand, his axiomatization is not entirely satisfactory from the standpoint of the formal simplicity and ease of comprehension that seem desirable or possible. For instance, the axiom on isotropy of sprays is quite complicated in its explicit form. There is a natural tension at work here between simplicity of formulation of axioms and their relative weakness. The Minkowski space-time of special relativity is a four-dimensional affine space. Schutz proves this, but it takes considerable development of the appropriate facts. In an effort to simplify the qualitative axioms for special relativity, I have come to feel that there is considerable justification in beginning with this assumption of an affine space, but admittedly it is easy to criticize this strong axiom on a number of grounds, including the extent to which it has a direct physical justification.

In stating that I see some justification in beginning with the assumption that the space is affine I have in mind a qualitative formulation for affine spaces based on the single primitive concept of betweenness for points. Another approach is to begin with finite dimensional real vector spaces, to specialize them to Minkowski spaces, and then to characterize the action of this vector space on the set of space-time points. A good discussion of this approach is to be found in Domotor [**1**]. My personal choice is for the kind of approach exemplified by Robb, Walker, and Schutz, but I think we still do not have the appropriately transparent set of qualitative axioms within this tradition.

Schutz gives a very thorough development of the consequences of his axioms, including a complete treatment of the Lorentz transformations. The order of development can be indicated by citing the chapter headings (in abbreviated form): introduction, kinematic axioms, conditionally complete particles, implications of collinearity, collinear sub-sprays after coincidence, collinear particles, theory of parallels, one-dimensional kinematics, three-dimensional kinematics, concluding remarks. Along the way, Schutz shows that each spray is a three-dimensional hyperbolic space with particles corresponding to "points" and with relative velocity as a metric function. In setting forth these various matters he uses the important characterization of elementary spaces due to Tits [**7**], [**8**] and Freudenthal [**3**], which brings his treatment of special relativity within the framework of one of the more important modern developments in the foundations of geometry. The comprehensiveness and up-to-date character of Schutz's treatment of the geometrical foundations of special relativity make it one of the better current references on the subject.

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Self-adjoint operators, by William G. Faris, Lecture Notes in Mathematics, No. 433, Springer-Verlag, Berlin, Heidelberg, New York, 1975, vi+115 pp., $7.80.

Quantum physics has greatly influenced the theory of self-adjoint operators throughout its development and continues to do so today. One problem arising in quantum physics, which is the main problem dealt with in the book under review, is the addition problem: When is the sum of two unbounded self-adjoint operators self-adjoint? More precisely, let A, B with domains $D(A)$, $D(B)$ be self-adjoint operators on a complex Hilbert space H. If the closure C of $A+B$ (defined on $D(A)\cap D(B)$) is self-adjoint, then we can regard C as "the" self-adjoint sum of A and B. More interesting are the cases in which C has many self-adjoint extensions, and the problem is to find the "right" one (if indeed there is a right one).

In quantum mechanics, kinetic and potential energy are described by self-adjoint operators, A, B say. Their sum is the total energy operator C, and to do quantum mechanics one must compute functions of it. One can do this (by the spectral theorem and the associated functional calculus) when C is self-adjoint. In particular, when C is self-adjoint, the dynamics of the system is described by the one parameter unitary group $\{\exp(-itC): t\in \mathbf{R}\}$, which is well defined. An example is the case of a spinless nonrelativistic quantum mechanical particle in a given potential. The Hilbert space is $H=L^2(\mathbf{R}^n)$ and the kinetic and potential energy operators are $A=-\Delta=-\sum_{j=1}^n \partial^2/\partial x_j^2$, $B=$the operator of multiplication by $V(x):\mathbf{R}^n\to\mathbf{R}$ ($B=V(x)$ for short). This set-up also describes two-body problems with no external potentials. The problem is to find the most general conditions on V so that $A+B$ (suitably interpreted) is self-adjoint.

One approach to the addition problem is via the Lie-Trotter product