9. C. L. M. H. Navier, *Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques*, Mem. Acad. Sci. Inst. France (2) **7** (1827), 375–393.

10. G. C. Sih, *Mechanics of fracture—1. Methods of analysis and solutions of crack problems*, Noordhoff, Leyden, 1973.

11. I. N. Sneddon and M. Lowengrub, *Crack problems in the classical theory of elasticity*, Wiley, New York and London, 1969. MR **41** #2986.

12. I. N. Sneddon, *Mixed boundary value problems in potential theory*, North-Holland, Amsterdam; Interscience, New York, 1966. MR **35** #6853.

13. C. Somigliana, Atti. Accad. Naz. Lincei Rend. **23** (1914), 463.

14. ———, Atti. Accad. Naz. Lincei Rend. **24** (1915), 655.

15. V. Volterra, *Sur l'équilibre des corps élastiques multiplement connexes*, Ann. Sci. École Norm. Sup. (3) **24** (1907), 401–517.

16. J. R. Willis, *Interfacial stresses induced by arbitrary loading of dissimilar elastic half-spaces joined over a circular region*, J. Inst. Math. Appl. **7** (1971), 179–197. MR **44** #3565.

17. ———. *The penny-shaped crack on an interface*, Quart. J. Mech. Appl. Math. **25** (1972), 367.

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Group theory and quantum mechanics, by B. L. van der Waerden, Die Grundlehren der math. Wissenschaften, Band 214, Springer-Verlag, Berlin, 1974, vii+211 pp.

In 1932 B. L. van der Waerden published *Die gruppentheoretische Methode in der Quantenmechanik.* Forty-two years later he published a translated and revised edition, *Group theory and quantum mechanics.* In the preface of the translated edition van der Waerden explains the intent of the original book and the reasons for a revision:

> Its aim was, to explain the fundamental notions of the Theory of Groups and their Representations, and the application of this theory to the Quantum Mechanics of Atoms and Molecules. The book was mainly written for the benefit of physicists who were supposed to be familiar with Quantum Mechanics. However, it turned out that it was also used by mathematicians who wanted to learn Quantum Mechanics from it. Naturally, the physical parts were too difficult for mathematicians, whereas the mathematical parts were sometimes too difficult for physicists.... In order to make the book more readable for physicists and mathematicians alike, I have rewritten the whole volume.

Before discussing whether van der Waerden has succeeded in his goal for the revised edition, let us briefly summarize the contents of the book. The book opens with Schrödinger's equation governing the state of a quantum mechanical system. Hilbert space is defined (as L^2 spaces only) and we are told a little about operators on Hilbert space. Some, but not all, of the details of the solution of the one electron atom (and, in particular, the hydrogen atom) are given. We meet the azimuthal, main, and magnetic quantum numbers and the terms of the spectroscopic series. Perturbation theory is touched upon, as is angular momentum, the normal Zeeman effect, and selection rules. This is all part of the explanation of the basics of quantum mechanics.

Next attention turns to groups and finite dimensional representations of groups on inner product spaces, with special emphasis on unitary representations. Lie groups and Lie algebras are introduced, with the focus on $SU(2)$, $O(3)$, $SL(2)$, and the restricted Lorentz group. By making use of the theory of invariants, van der Waerden determines all irreducible finite dimensional representations of these groups. There is occasional mention of how the notions of group theory will prove useful in quantum mechanics and there is a fairly detailed discussion of selection and intensity rules. The theory of the representations of the symmetric group on n elements, however, is deferred until much later in the book. It appears after the discussion of permutations of the electrons in a system.

The first topic in physics which van der Waerden treats after having disposed of the fundamentals of group representations is the electron and its spin. There is a very brief discussion of the physical evidence for spin and the requisite hypotheses for a quantized spin angular momentum. Spin coordinates are added to the wave function and the doublet splitting of the alkali terms is deduced using representations of the orthogonal group. Dirac's relativistic wave equation for a single electron is discussed and brief mention is made of two other elementary particles, the positron and the neutrino. The case of several electrons is handled nonrelativistically and the terminology of spectroscopy is introduced. Other topics considered include the Pauli exclusion principle, the periodic system, and various aspects of atomic spectroscopy. While in general van der Waerden uses group theoretic methods, he does present Slater's method for determining which terms will appear in the spectrum of an element.

The book closes with a discussion of molecular spectra. Van der Waerden confines himself to the case of molecules with two nuclei and determines the quantum numbers and the selection rules for such molecules. Some topics, such as the banding phenomena in the spectra of molecules and the stability of molecules, are discussed mostly qualitatively.

Now let us turn to the question of whether van der Waerden has met the objectives set forth in his preface. It is probably futile for a mathematician to speculate on how much a physicist would benefit from a book; in this case it is also academic. Most physicists will undoubtedly continue to turn to Eugene Wigner's *Group theory*. Wigner's book, first published in German in 1931, and then in translation, revised and expanded, in 1959, is considerably longer than van der Waerden's book and much fuller in detail. Some mathematicians, on the other hand, might well prefer a book which explains the essentials of quantum mechanics and the applications of group theory without presenting a complete and detailed picture. Van der Waerden's book promises to meet this need; but, unfortunately, it fails.

The tenor of the whole book is set at the very outset: we are told that a pure state of a mechanical system is defined by a wave function which is a solution of Schrödinger's differential equation. We are not told what a pure state is physically; the statistical interpretation of the wave function is given only three lines; and no attempt whatsoever is made to connect Schrödinger's equation with physical reality. This spirit repeats again and again throughout the book; the physics is assumed, not explained. A mathematician who does not already know quantum mechanics will be unable to read this book without extensive supplementary readings. The reader of this book obtains no sense of the physical reality of the subject discussed. This is a bit ironic when one considers the heavy emphasis on spectroscopy in van der Waerden's book, for it was precisely spectroscopy which provided some of the best experimental evidence for the theory of quantum mechanics.

When we are introduced to the spectroscopic terms and the terminology for them we can see a connection with representations of the orthogonal group, but not with anything that we might see in the laboratory. There are a number of term diagrams scattered throughout the book, but they have little connection with the text and there is almost no explanation how to use them. It would, at least, have been a nice touch to explain the sequence of letters, $S, P, D, \cdots$, used to designate spectroscopic series. (The lines in the S series are usually quite sharp; the most intense lines of the spectrum lie in the P or principal series; and the lines of the D series are generably rather diffuse.)

I should make some comments on the purely mathematical portions of the book (nearly half the volume). The mathematical treatment is reasonable, though somewhat old-fashioned. The range of topics discussed is limited, as is appropriate to a book directed primarily to applications to physics. This means, of course, that this book is not suitable for a reader interested primarily in representation theory. At various locations in the book, as appropriate, revision has taken into account what we have learned since 1932. But nowhere does van der Waerden take into account what we have forgotten. As mentioned earlier, the representations of certain of the classical groups are described in the language of the theory of invariants. In 1932 most mathematicians were undoubtedly quite familiar with this subject, which had been an area of active research up to the time Hilbert proved his basis theorem. Today there are probably few who feel completely comfortable with this theory. (How many of us have even heard of Gordan, the "King of Invariants"?) Although van der Waerden does provide some explanation of the techniques, it is not adequate. The only reference he gives us is to a book published in 1903! While van der Waerden does show the general power of the theory of invariants for determining the finite dimensional representations of a group of matrices, there are some results that are much easier than as presented in his book. To cite one example, van der Waerden represents the (complex) group $SL(2)$ as the restricted Lorentz group in the following fashion (and with the following notation): let $\overset{1}{u}$, $\overset{2}{u}$, $\overset{\dot 1}{u}$, and $\overset{\dot 2}{u}$ be indeterminates and let the element $\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ in $SL(2)$ act on these two pairs of indeterminates by the formulae:

$$\overset{1}{u}' = \overset{1}{u}\alpha + \overset{2}{u}\gamma, \qquad \overset{2}{u}' = \overset{1}{u}\beta + \overset{2}{u}\delta$$

and

$$\overset{\dot 1}{u}' = \overset{\dot 1}{u}\alpha^* + \overset{\dot 2}{u}\gamma^*, \qquad \overset{\dot 2}{u}' = \overset{1}{u}\beta^* + \overset{\dot 2}{u}\delta^*.$$

(* indicates complex conjugation.) These transformations leave invariant the space of all Hermitian bilinear forms,

$$c_{11\cdot}\overset{11\cdot}{u\dot{u}} + c_{12\cdot}\overset{12\cdot}{u\dot{u}} + c_{21\cdot}\overset{21\cdot}{u\dot{u}} + c_{22\cdot}\overset{22\cdot}{u\dot{u}},$$

(where $c_{11\cdot}$ and $c_{22\cdot}$ are real and $c_{12\cdot}$ and $c_{21\cdot}$ are complex conjugates). Furthermore, the determinant $c_{11\cdot}c_{22\cdot} - c_{12\cdot}c_{21\cdot}$ is also left invariant by these transformations. A change of variables,

$$c_{21\cdot} = x + iy, \quad c_{12\cdot} = x - iy, \quad c_{11\cdot} = z + ct, \quad c_{22\cdot} = -z + ct,$$

enables one to see that we are representing $SL(2)$ as real Lorentz transformations. The verifications of the facts above and others not cited are straightforward, but exceedingly laborious. The exact same representation is obtained if we let V be the (real) vector space consisting of selfadjoint 2×2 matrices and let ρ be the representation of $SL(2)$ acting on V defined by $\rho(A)(C) = ACA^*$, for all $A \in SL(2)$ and $C \in V$. (Here * denotes the usual adjoint, i.e. the conjugate transpose of a matrix.) The tedious calculation that the determinant is left invariant, which van der Waerden, of course, omits, is now nothing more than an immediate consequence of the familiar rule for the determinant of a product. Indeed, every detail required by van der Waerden becomes a simple exercise in linear algebra.

There are a number of minor faults about the book which we would gladly overlook if the main objectives were achieved. Many of the bibliographic references are out of date and all are scattered through the book in footnotes. This makes it hard to find a citation if one does not remember where to look. There are extraordinarily many misprints, enough to be really annoying. Most of them matter little; but some are vital to the meaning, especially the ones appearing in displayed formulae. These, at least, should have been proofread carefully. Equally annoying is van der Waerden's habit of using symbols without telling us what they mean. Most of them are standard, and so present little difficulty to the reader familiar with quantum mechanics; but if one happens, for example, not to realize that the Z in the Coulomb potential, Ze/r, for a single electron in a central field refers to the atomic number of the nucleus, then the meaning is lost. Similar problems may confront the reader unacquainted with Gordan's second order wave equation, which is offered as motivation for Dirac's relativistic wave equation for an electron.

Thus, as we have seen, the mathematician who wishes to learn quantum mechanics, and take advantage of his mathematical knowledge in doing so, must look elsewhere. The mathematician familiar with quantum mechanics and desirous of seeing how group theory can be applied to physics may benefit from and enjoy van der Waerden's book. However, the physics described by van der Waerden, atomic and molecular spectroscopy, can be done for the most part without group representations. It is in other subjects, such as elementary particles, where the dynamics is mostly unknown, and where, as a consequence, we must rely heavily on conservation laws and symmetries, that group theory has a truly vital role to play. And so the mathematician who already knows quantum mechanics will likely be much more excited by a book such as D. B. Lichtenberg's *Unitary symmetry and elementary particles.* This book, incidentally, contains a lovely explanation of the relationships among conservation laws, symmetries, and group representations. These points are virtually untouched by van der Waerden. For that matter, I might mention that $SU(n)$ for $n \geqq 3$ and, in particular, $SU(3)$, groups of considerable importance in modern physics, are totally absent from van der Waerden's book.

For the mathematician who would like to learn about quantum mechanics for the first time, I would recommend the Feynman *Lectures on physics.* As is well known, the third volume of these lectures achieves a remarkable tour de force: quantum mechanics is presented with only the most elementary use of mathematics. It would be nice to have, in addition, a book complementary to Feynman's a book which assumes considerable mathematical knowledge and maturity and yet which presents quantum mechanics with a minimum of physical prerequisities, but with the same force and sense of reality as in the Feynman lectures.

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Jean Dieudonné,[1] *Cours de géométrie algébrique.* Vol. 1: *Aperçu historique sur le développement de la géométrie algébrique,* 234 pp.; Vol. 2: *Précis de géométrie algébrique élémentaire,* 222 pp., Collection SUP, Presses Universitaires de France, Paris, 1974, paperback, pocketbook size. Each vol. 34.88F in France = approx. $8.30.

I. R. Shafarevich,[2] *Basic algebraic geometry,* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band 213, Springer-Verlag, Berlin, 1974, xv+439 pp., $40.20. (Translated from the Russian by K. A. Hirsch)

The author of an introductory book on algebraic geometry faces many difficult choices. How is he to introduce his reader to some of the basic examples of algebraic geometry, give him some motivation, and teach him the modern language of the subject? As Dieudonné says in his introduction, "Algebraic geometry is surely that branch of mathematics having the greatest gap between the intuitive ideas which form the point of departure and the complex abstract concepts which lie at the base of modern research." No introductory book will succeed unless it makes a serious attempt to bridge this gap.

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[1] Dieudonné's Volume 1 is an expanded version of an article *The historical development of algebraic geometry,* Amer. Math. Monthly **79** (1972), 827–866.

[2] The first four chapters of Shafarevich's book are almost identical with an earlier book, which has appeared in English translation as *Foundations of algebraic geometry,* Russian Math. Surveys **24** (1969), 1–178.