treatment than the rest. One feels this not so much because of the relative amount of space devoted to the various subjects but rather because these 'set-theoretical' results are more final and self-sufficient in character than the others, some of which sometimes appear to be somewhat technical and not as much justified in themselves.

Many things (Keisler's two-cardinal theorem, Morley's "Hanf-number" theorem on omitting types) could have been put in their true contexts only in extensions of first order model theory (generalized quantifiers, infinitary logic).

A particular matter that should have received more attention in the book is Fraissé-Ehrenfeucht games (and some generalizations). These are treated only in exercises. These games are important, particularly through the work of Lindström who applied them to give a theory of preservation theorems ("regular relations") (Theoria **32** (1966), 171–185), and to his celebrated work on characterizing first order logic.

It should be added to the discussion of transcendence rank that the rank of a formula defined in a not necessarily ω_1-saturated model is simply taken to mean rank in any (cf. Lemma 7.1.20) ω_1-saturated elementary extension. Then the first sentence of the proof of 7.1.23 can be deleted, and it should be because as it stands it is incorrect. Furthermore, the proof of 7.1.23 uses the fact that α is regular (and Victor Harnik tells us that the theorem is false without this assumption).

The proof of 7.2.2 is written up in a somewhat awkward way, and in fact, the induction hypothesis (4) is not stated correctly. In the proof of 7.3.7, the definition of the structure A was omitted (but can be guessed). On p. 480 the numerical code (1) should be shifted to the next displayed formula.

In the review, the reviewer could not bring himself to suppressing the use of the word "structure" in favor of the word "model" as it is done in the book.

In conclusion, let us say that in this book model theory has received a thoroughly worthy exposition that will no doubt help establish the deserved status of model theory as an original, rich, useful and mature branch of mathematics.

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Mathematical theory of dislocations and fracture, by R. W. Lardner, University of Toronto Press, Toronto, 1974, xi+363 pp.

The mathematical theory of elasticity has a rich and varied history. It is concerned with the mathematical study of the response of elastic bodies to the action of forces. There is no doubt that the linear theory is one of the more successful theories of mathematical physics. A beautiful account of this theory is found in Gurtin (1970).

The first attempt to set the elasticity of bodies on a scientific foundation was undertaken by Galileo and is described in his Discourses, published in

1638. The elasticity properties are characterized by certain functions relating forces to deformations. In this regard, Hooke's law (1678) is of fundamental importance. His law states, in effect, that the extension of springlike bodies, produced by the tensile forces, is proportional to the forces. This law really forms the basis of the mathematical theory of elasticity. In the interval of time between the discovery of Hooke's law and that of the general differential equations of elasticity by Navier, about 1821, research was chiefly directed toward the solution and extension of Galileo's problem, the related theories of the vibrations of bars and plates, and the stability of columns. Significant contributions during this period were made by Jacob and Daniel Bernoulli, Euler, and Coulomb. Navier's memoir (1827) marks the birth of the mathematical theory of elasticity. His work attracted the attention of Cauchy (1828) who, working from different assumptions, gave a formulation of the linear theory of elasticity that remains virtually unchanged to the present day. The next major contribution was that of George Green who in 1837 deduced the basic equations of elasticity from the principle of virtual work. The theoretical development of the subject flourished until the early twentieth century with the work of Beltrami, Bette, Boussenesq, Kelvin, Kirchoff, Lamé, Saint-Venant, Somigliana, Stokes, and others. The basic theorems of compatibility, reciprocity, and uniqueness were established by these authors. In addition, they derived important general solutions of the underlying field equations. The twentieth century saw a shift in emphasis to the solution of boundary value problems, and the theory remained relatively dormant until the middle of the twentieth century. Then new results appeared concerned with, among other things, Saint-Venant's principle, stress functions, variational principles, and uniqueness.

During recent years a large number of new books has appeared which, in company of the older classics such as Love (1892), provides an excellent comprehensive treatment of the subject, and a basis for the understanding and evaluation of the steady stream of research papers. The most notable of this list are Lekhnitzki (1950), Savin (1951), Muskhelishvili (1953), Galin (1953), Green and Zerna (1954), and Knops and Payne (1971). The preponderance of Russian titles, both for books and research articles, is striking. The importance of the elegant work on the application of function theoretic methods to two dimensional problems in elasticity of Muskhelishvili and his school in Tbilisi cannot be overstressed.

Among the more active areas of research concerned with special applications of the mathematical theory are the subjects of dislocation and fracture. A dislocation is the principal type of defect that influences the mechanical properties of crystals. Elastic deformations in a crystal may arise not only by the action of external forces on it but also because of internal structural defects present in the crystal. For the purpose of a macroscopic discussion, we can regard a crystal as a continuous medium. Dislocations are the type of defects produced when cuts are made in the medium, the two sides of the cut are displaced relative to each other (material being added or removed as necessary), and the cuts rewelded. These defects were first studied by Volterra (1907) and Somigliana (1914, 1915). Their main concern was with certain deformations of an elastic body in which the displacement field is not single-valued. The dislocation deformation has the following general property: let L denote any closed contour which encloses the dislocation line D (see Figure 1). After a passage around L, the elastic displacement vector $\boldsymbol{u}$ receives a certain finite increment $\boldsymbol{b}$ (i.e., $\boldsymbol{b}=\boldsymbol{u}_{\mathrm{final}}-\boldsymbol{u}_{\mathrm{initial}}$, the change in displacement between the initial and final points on the path). The vector $\boldsymbol{b}$ is called the Burgers vector. The above property can be expressed by,

$$\oint_L du_i = \oint_L \frac{\partial u_i}{\partial x_j}\, dx_j = -b_i \tag{1}$$

where u_i, b_i are the components of the displacement vector, and the Burgers vector, respectively. The direction in which the contour is traversed and the chosen direction of the tangent vector $\boldsymbol{\tau}$ to the dislocation line are assumed to be related by the corkscrew rule (see Figure 1). The dislocation line is itself a line of singularities of the deformation field. If $\boldsymbol{\tau}$ is perpendicular to $\boldsymbol{b}$ the dislocation is called an edge dislocation, while if $\boldsymbol{\tau}$ is parallel to $\boldsymbol{b}$ it is called a screw dislocation. Condition (1) states the essential fact that in the presence of a dislocation the displacement vector is *not* a single-valued function. The main problem is to determine the effect of dislocations on the stress field. For an isotropic medium, this requires an investigation into the solutions of the differential equation

$$\Delta \boldsymbol{u} + \frac{1}{1-2\eta}\,\mathbf{grad}\ \mathrm{div}\ \boldsymbol{u} = \boldsymbol{\tau}\otimes \boldsymbol{b}\delta(\xi)$$

subject to appropriate boundary conditions. Here $\boldsymbol{u}$ represents the dislocation, η Poisson's ratio of the material and $\boldsymbol{\tau}$, $\boldsymbol{b}$ as above. A review of the literature in this area up to 1967 is contained in a very nice article by Bilby and Eshelby (1968).

Dislocations enter the theory of fracture in several ways. First, as crystal dislocations, they play a role in the physics of fracture. Secondly, they can

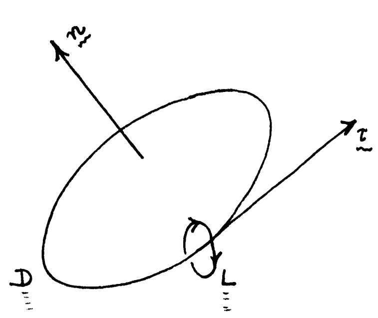

FIGURE 1

serve as a convenient "mathematical element" in the macroscopic treatment of fracture. This is a result of the fact that a crack is equivalent to a continuous array of dislocations. One can utilize this equivalence to develop the mathematical theory of dislocation arrays and cracks. We can think of a crack as a discontinuity in the displacement vector. Interest in crack problems in the mathematical theory of elasticity arises from the theory of brittle fracture, which itself originated nearly fifty years ago in the classical work of Griffith (1921). Since the number of materials that fail under normal conditions in a brittle fashion is relatively small, this theory was regarded as of academic rather than practical interest for many years, more as a source of interesting mixed boundary value problems (Sneddon, 1966) than as a growing part of solid mechanics. Interest has been revived in the theory in recent years as a result of the experimental discovery that at high or low temperatures many structural elements composed of commonly used materials that display plastic properties in standard tensile tests fail by a "quasi-brittle" process. By this we mean that failure occurs by the propagation of cracks and that although there is a plastic zone, it is of limited extent and concentrated at the crack tip. Surveys of research in the mathematical aspects of this subject are contained in Sneddon and Lowengrub (1968), Goodier (1968), and Lih (1973). In recent years a steady stream of articles has appeared in the literature, but the most notable work (concerning the mathematical theory) is that of Willis (1971, 1972) and Knowles and Sternberg (1973, 1974). Willis solves the appropriate boundary value problems by first setting up and solving an integral equation for the Radon transform of the relative displacement of the crack faces. His formulation unifies a large part of the literature on elastostatics and elastodynamics. Knowles and Sternberg consider an asymptotic treatment, consistent with the nonlinear equilibrium theory of compressible elastic solids, of the stresses and deformations near the tip of a traction-free crack in a slab of all around infinite extent under conditions of plane strain. The asymptotic analysis of the problem is reduced to a nonlinear eigenvalue problem. The solution of this problem is established in closed form in terms of elementary functions and a transcendental integral of such functions. Their results are of extreme importance in the development of the theory of crack problems beyond the scope of classical elasticity.

It is appropriate at this stage also to include a brief discussion of the uniqueness question for domains containing cracks. Although boundary value problems in linear elastostatics for domains containing cracks have received considerable attention, the uniqueness of solutions to these problems is not guaranteed by the standard uniqueness theorem due to Kirchoff. This theorem states (see Knops and Payne (1971) for an exposition of such theorems) that *there is at most one solution to the isotropic standard boundary value problems of plane strain provided* $-\infty < \nu < \frac{1}{2}$, *where* ν *is Poisson's ratio and* $\nu \neq 0$. *In the traction boundary value problem there is uniqueness to within a rigid body displacement.* The inapplicability of this theorem to crack problems arises from the presence in their solutions of singularities at the crack tips, so that the smoothness ordinarily assumed in proving uniqueness is lacking. Knowles and Pucik (1973) prove the uniqueness (in two dimensions) of the solution to the traction boundary value problem in linear elastostatics for a bounded domain containing a crack. One would hope that this can open the door to uniqueness theorems for other boundary value problems for domains containing cracks. Their proof does require the displacement field to be bounded near the crack tips.

The book by Lardner is an attempt to describe those aspects of dislocation theory which are closely related to the theories of elasticity and macroscopic plasticity, to modern continuum mechanics, and to the theory of cracks and fracture. It is intended for students and research workers in both mechanics and applied mathematics. It is disappointing that this text, although titled, *The mathematical theory of dislocations and fracture*, does not consider any questions of uniqueness. Most of the book is concerned with special problems. No mention is made of the work of Willis, or Knowles and Sternberg described above.

More specifically, the first three chapters of the text cover the basic material. The style in these chapters seems uneven and students would certainly have difficulty in viewing the subject as a coherent entity. As far as the remaining chapters are concerned, the eighth chapter is the most interesting. This chapter poses and solves the internal stress problem for linear isotropic materials. The text does include a very good set of references at the end of each of the chapters.

The entire subject of the theory of fracture needs careful mathematical attention. One need only look at the abundance of papers in both engineering and applied mathematics journals for verification. Unfortunately, this book is not going to attract the interest of young applied mathematicians. Books written on the level and style of Gurtin (1970) will provide a far greater incentive for such investigations.

REFERENCES

1. B. A. Bilby and J. D. Eshelby, *Dislocations and the theory of fracture*, Fracture, vol. I, H. Liebowitz (editor), Academic Press, New York and London, 1968.

2. A. L. Cauchy, *Sur les équations qui expriment les conditions d'équilibre on les lois du mouvement intérieur d'un corps solide, élastique ou non élastique*, Ex. Math. **3** (1828), 160–187; also in œuvres (2), 8, 195–226.

3. J. N. Goodier, *Mathematical theory of equilibrium cracks*, Fracture, vol. II, H. Liebowitz (editor), Academic Press, New York and London, 1968.

4. A. A. Griffith, *The phenomena of rupture and flow in solids*, Philos. Trans. Roy Soc. London Ser. A **221** (1921), 163–198.

5. M. E. Gurtin, *Linear theory of elasticity*, 2nd ed., Handbuch der Physik, vol. VI, Springer-Verlag, Berlin, Heidelberg and New York, 1970.

6. J. K. Knowles and T. A. Pucik, *Uniqueness for plane crack problems in linear elastostatics*, J. Elasticity **3** (1973), 155–160.

7. J. K. Knowles and E. Sternberg, *An asymptotic finite deformation analysis of the elastostatic field near the tip of a crack*, J. Elasticity **3** (1973), 67–107.

8. ———, *Finite-deformation analysis of the elastostatic field near the tip of a crack: Reconsideration and higher-order results*, J. Elasticity **4** (1974), 201–233.

9. C. L. M. H. Navier, *Mémoire sur les lois de l'équilibre et du mouvement des corps solides élastiques*, Mem. Acad. Sci. Inst. France (2) **7** (1827), 375–393.

10. G. C. Sih, *Mechanics of fracture—1. Methods of analysis and solutions of crack problems*, Noordhoff, Leyden, 1973.

11. I. N. Sneddon and M. Lowengrub, *Crack problems in the classical theory of elasticity*, Wiley, New York and London, 1969. MR **41** #2986.

12. I. N. Sneddon, *Mixed boundary value problems in potential theory*, North-Holland, Amsterdam; Interscience, New York, 1966. MR **35** #6853.

13. C. Somigliana, Atti. Accad. Naz. Lincei Rend. **23** (1914), 463.

14. ———, Atti. Accad. Naz. Lincei Rend. **24** (1915), 655.

15. V. Volterra, *Sur l'équilibre des corps élastiques multiplement connexes*, Ann. Sci. École Norm. Sup. (3) **24** (1907), 401–517.

16. J. R. Willis, *Interfacial stresses induced by arbitrary loading of dissimilar elastic half-spaces joined over a circular region*, J. Inst. Math. Appl. **7** (1971), 179–197. MR **44** #3565.

17. ———. *The penny-shaped crack on an interface*, Quart. J. Mech. Appl. Math. **25** (1972), 367.

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Group theory and quantum mechanics, by B. L. van der Waerden, Die Grundlehren der math. Wissenschaften, Band 214, Springer-Verlag, Berlin, 1974, vii+211 pp.

In 1932 B. L. van der Waerden published *Die gruppentheoretische Methode in der Quantenmechanik.* Forty-two years later he published a translated and revised edition, *Group theory and quantum mechanics.* In the preface of the translated edition van der Waerden explains the intent of the original book and the reasons for a revision:

> Its aim was, to explain the fundamental notions of the Theory of Groups and their Representations, and the application of this theory to the Quantum Mechanics of Atoms and Molecules. The book was mainly written for the benefit of physicists who were supposed to be familiar with Quantum Mechanics. However, it turned out that it was also used by mathematicians who wanted to learn Quantum Mechanics from it. Naturally, the physical parts were too difficult for mathematicians, whereas the mathematical parts were sometimes too difficult for physicists.... In order to make the book more readable for physicists and mathematicians alike, I have rewritten the whole volume.

Before discussing whether van der Waerden has succeeded in his goal for the revised edition, let us briefly summarize the contents of the book. The book opens with Schrödinger's equation governing the state of a quantum mechanical system. Hilbert space is defined (as L^2 spaces only) and we are told a little about operators on Hilbert space. Some, but not all, of the details of the solution of the one electron atom (and, in particular, the hydrogen atom) are given. We meet the azimuthal, main, and magnetic quantum numbers and the terms of the spectroscopic series. Perturbation theory is touched upon, as is angular momentum, the normal Zeeman effect, and selection rules. This is all part of the explanation of the basics of quantum mechanics.