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BOOK REVIEWS

Topics in operator theory, by C. Pearcy, Mathematical Surveys, No. 13, Amer. Math. Soc., Providence, Rhode Island, 1974, 235+ix pp., $23.00.

Topics in operator theory consists of five diverse expository papers surveying a number of recent and not so recent ideas in the theory of bounded linear operators on Hilbert space. The essays range from an exposition of multiplicity theory for normal operators by Arlen Brown to an inclusive and detailed review of the current state of our knowledge of weighted shift operators by Allen L. Shields. Also one finds an illuminating introduction to invariant subspaces and their relation to various problems in analysis by Donald Sarason, a concise geometric introduction to the model theory of Sz.-Nagy and Foiaş by R. G. Douglas and finally a report on the recent powerful technique of V. I. Lomonosov for producing invariant subspaces of operators related to compact operators by Carl Pearcy and Allen L. Shields.

Two results concerning operators on finite dimensional complex vector spaces lead one to feel that one has a fairly good grasp of their structure. First, every operator can be represented by an upper triangular matrix with respect to some orthonormal basis; and second, if arbitrary bases are admitted, then every operator can be represented by a matrix in Jordan normal form.

One of the principal goals of operator theory is to obtain a comparable understanding of the structure of operators on arbitrary Hilbert or Banach spaces. The invariant subspace problem is an indication of just how far we are from attaining such a goal. For a trivial consequence of either of the two results on finite dimensional spaces is that every operator on a space $\mathcal{H}$ of dimension at least two has a nontrivial invariant subspace, a subspace other than (0) or $\mathcal{H}$ that is mapped into itself. Whether or not every operator on an infinite dimensional space has a closed invariant subspace remains an open question despite intense effort in the past few years.

With the general problem so intractable the focus naturally shifts to special classes of operators. In the book under review we find an examination of several of these, some illustrations of how they relate to other parts of analysis, and an introduction to one attempt at a general model theory, which the special cases have led to. The preparation required of the reader varies a bit between the articles, but generally requires the standard courses on measure theory, complex function theory and functional analysis. As the audience will probably differ somewhat from essay to essay, we will comment on this as we take up each in order of appearance. In general the book provides a good introduction to some (but by no means all) of the topics which have interested operator theorists in the past few years.

The first article, *Invariant subspaces*, by Donald Sarason, deals mainly with operators tied to the unit circle of the complex plane and related problems in analysis. Starting out somewhat more generally, he determines the reducing subspaces (those invariant subspaces with an invariant orthogonal complement) of a special class of normal operators and indicates how one obtains the corresponding results for arbitrary normal operators. The special case is that of "multiplication by z", M_z, on the L^2 space of a Borel measure μ supported on a compact subset of the plane: $(M_z f)(\lambda) = \lambda f(\lambda)$. These operators have a special significance since they form the building blocks for the most general normal operator. The operators that commute with M_z are the multiplication operators M_ϕ determined by functions ϕ in $L^\infty(\mu)$, and the reducing subspaces are just the ranges of those M_ϕ determined by characteristic functions. A reflexivity theorem for certain algebras of such multiplication operators asserts that these algebras are determined by their invariant subspaces in the sense that the only operators leaving invariant all the invariant subspaces of the algebra are the operators in the algebra. This reflexivity result of the authors is later shown to lead to an approximation theorem and to Wermer's maximality theorem.

When μ is specialized to a measure supported on the unit circle, M_z becomes unitary. For arbitrary unitary operators the irreducible or pure invariant subspaces (those containing no nontrivial reducing subspace) have a particularly simple geometric description. If U is unitary, then a wandering subspace for U is a subspace that is orthogonal to its images under powers of U. Every irreducible invariant subspace corresponds to a wandering subspace $\mathscr{L}$ and is the direct sum of $\mathscr{L}$ and its images under all positive powers of U. A consequence turns out to be the classical F. and M. Riesz theorem on analytic measures.

When μ is specialized still further to Lebesgue measure, $d\theta \mid 2\pi$, on the circle, M_z becomes the bilateral shift. (Examining the action of M_z on the orthonormal basis $\{z^n : n = 0, \pm 1, \pm 2, \cdots\}$ makes its name manifest.) The determination of the irreducible invariant subspaces of M_z leads to Beurling's theorem characterizing the invariant subspaces of the unilateral shift, which may be obtained by restricting M_z to H^2. (The set $\{z^n : n = 0, 1, 2, \cdots\}$ is a basis for H^2; hence the term "unilateral".) The invariant subspaces of U all take the form ϕH^2 where ϕ is an inner function, i.e. an H^2 function with unit modulus a.e. Part of the significance of Beurling's result was due to the existence of a complete description of inner functions of which Sarason obtains as much as possible by examining the invariant subspaces.

Another theme interwoven in this article is that of the invariant subspaces of one-parameter semigroups of unitary operators. Specialization to the group of translations on $L^2(-\infty, \infty)$ leads via Fourier transform and a conformal mapping to a surprising and beautiful connection with the unilateral shift operator. A result of this connection is the use of Beurling's theorem to arrive at a description of the invariant subspaces of the Volterra operator: $Vf(x) = \int_0^x f(t)\, dt$ for f in $L^2(0, 1)$. That these subspaces are all of the form $L^2(a, 1)$ for $0 \leqq a \leqq 1$ was originally arrived at by means of the Titchmarsh convolution theorem. Here we see that Titchmarsh theorem derived as a consequence of the invariant subspace result.

To climax a well-written exposition of known facts, Sarason derives a new result by constructing an operator whose lattice of invariant subspaces is isomorphic to the lattice of closed subsets of the unit interval. The article can be recommended to any student who has had some exposure to Hilbert spaces and to anyone curious about all the fuss over the invariant subspace problem. There is more involved than just the challenge of an unsolved problem.

The unilateral shift can also be defined abstractly as the operator that shifts a basis: if $\{e_n: n=0, 1, 2, \cdots\}$ is an orthonormal basis let $Ue_n = e_{n+1}$. An immediate generalization is obtained by taking a bounded sequence of complex numbers w_n and setting $Te_n = w_n e_{n+1}$. This defines a unilateral *weighted* shift. Had the basis been indexed by all the integers instead, we would have arrived at the definition of a *bilateral* weighted shift. In the second article, *Weighted shift operators and analytic function theory*, Allen L. Shields puts together virtually everything known about these operators and presents us with a lucid survey of the subject which will be useful to the expert as well as the beginner. Included are a bibliography of 106 items, a section carefully delineating credits for the results, and thirty-three questions for further research. Of the latter, several were answered while the book was in preparation and the answers were added in proof.

The article begins with the unitary equivalence and similarity results from R. L. Kelley's thesis. Changing the arguments of the weights w_n gives a unitarily equivalent weighted shift, so attention may be restricted to the case of nonnegative weights. Since most of the interest centers on shifts with nonzero weights, this further restriction is also made. Now the transition may be made to the equivalent and fruitful point of view in which a weighted shift appears as "multiplication by z", M_z, on a space of formal power series. Such a space consists of all series $\sum \alpha_n z^n$ satisfying $\sum |\alpha_n|^2 \beta(n)^2 < \infty$, where $\{\beta(n)\}$ is a fixed sequence of positive numbers, which is easily related to the original weight sequence $\{w_n\}$. The range of n is the set of nonnegative integers in the unilateral case and the set of all integers in the bilateral. (The bipartite nature of this theory soon becomes evident. To a great extent the article consists of a set of pairs of results with a unilateral one followed by an appropriate bilateral modification or vice versa.) The parallel with the original unweighted H^2 or L^2 space being evident, the space on which the weighted shift acts is called $H^2(\beta)$ or $L^2(\beta)$ respectively.

From the new standpoint many things are clearer. For example, the operators that commute with M_z appear as "multiplication operators" M_ϕ, which leads to the introduction of the space of multipliers $H^\infty(\beta)$ or $L^\infty(\beta)$ respectively. A great deal of information appears about the shape, size and parts of the spectrum. When it is not too small, function theoretic techniques permit deeper analysis. Classical results concerning H^2 and H^∞ suggest theorems about weighted shifts as well as some fascinating unsolved problems. One of the more striking ones is a corona conjecture for $H^\infty(\beta)$.

Other sections delve into criteria for hypo- and subnormality and for strict cyclicity of algebras generated by weighted shifts. A section is devoted to invariant subspaces of weighted shifts and, in particular, to conditions for unicellularity of unilateral weighted shifts. (An operator is called *unicellular* if its invariant subspaces are linearly ordered by inclusion.) Whether or not a bilateral weighted shift can be unicellular is an intriguing open problem. The exposition in this admirable article concludes with a section on cyclic vectors.

On spaces of finite dimension far more can be said about normal operators than the two general structure results stated earlier. First, every normal operator has a basis of eigenvalues, and second, two normal operators are unitarily equivalent if and only if they have the same eigenvalues and corresponding eigenspaces have the same dimension. The former result finds its infinite dimensional generalization in the spectral theorem and the latter in multiplicity theory, which Arlen Brown describes in the third article, *A version of multiplicity theory.*

As alluded to earlier, the general normal operator on a space of arbitrary dimension (separable or not) may be assembled by forming a direct sum of multiplication operators M_z on spaces $L^2(\mu)$. How to tell when two such assemblages are unitarily equivalent requires an examination of σ-ideals of measures, according to an approach first taken by Wecken in 1939, and this examination is carried out in very readable detail in the article. A multiplicity theory for separable spaces, developed by Hellinger, had preceded Wecken by some 32 years. The special case, in which the requisite σ-ideals correspond to Borel sets in the plane, is shown to follow naturally as a corollary to the general development.

In this article students can find a straightforward clean approach to a multiplicity theory valid on nonseparable spaces. My only quibble is with the author's choosing to omit mention of the Fuglede theorem at the time the spectral theorem is stated.

It is commonplace that the unilateral shift is the typically infinite dimensional operator in that it is almost invariably the operator to consider when one wishes to exhibit a distinction between finite and infinite dimensional behavior. Just how fundamental it is became apparent in 1959 when Rota showed that one can obtain a similarity model for every operator whose spectrum lies in the open unit disc by compressing a direct sum of infinitely many copies of the shift to the orthogonal complement of an appropriately chosen one of its invariant subspaces. A modification of Rota's construction by de Branges and Rovnyak actually produced unitary equivalence models for contractions T (i.e. operators with $\|T\| \leqq 1$) such that powers of T^* tend strongly to zero. Another approach which led to the same conclusion was the attempt to understand contractions as "parts" of unitary operators. In 1950 P. R. Halmos showed that if T is a contraction on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ can be embedded in a larger space $\mathcal{K}$ on which there is defined a unitary operator W such that T is the *compression* of W to $\mathcal{H}$, i.e. $T = PW \mid \mathcal{H}$, where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. B. Sz.-Nagy showed three years later that W can be chosen to also satisfy $T^n = PW^n \mid \mathcal{H}$

for nonnegative n. Such a W, called a *unitary dilation* or T, is unique up to unitary equivalence if it is minimal in the sense that no restriction of W to one of its invariant subspaces including $\mathcal{H}$ is unitary. The elucidation of the relations between W and T constitutes one of the major tasks of R. G. Douglas' article, *Canonical models.*

The article begins with the construction of the minimal unitary dilation of a contraction. It is shown that if W is a unitary operator on a space $\mathcal{K}$ and if T is the compression of W to a subspace $\mathcal{H}$ then W is a dilation of T in the above sense if and only if the orthogonal complement of $\mathcal{H}$ may be decomposed as a direct sum of an invariant subspace of W^* with an invariant subspace of W. The subspace $\mathcal{H}$ is then called semi-invariant, and U is the minimal unitary dilation of T only if these two invariant subspaces are irreducible. As mentioned earlier, the irreducible invariant subspaces of unitary operators may be completely described, and when this is done in the context of a direct integral representation of $\mathcal{K}$ and W, the characteristic operator function Θ of Sz.-Nagy and Foiaş and the resulting functional model for the contraction T emerge in a very smooth manner. Included is a brief discussion of the relation between invariant subspaces of T and factorizations of Θ. This relation was the basis for a seemingly promising attempt on the general invariant subspace problem about a decade ago. Unfortunately there is an apparently insurmountable obstacle to this approach, which is pointed out in the article.

Another line of investigation in Douglas' paper begins with the subtle and powerful extension of the Riesz functional calculus by Sz.-Nagy and Foiaş. For each contraction T having no unitary direct summand this extension yields a natural homomorphism from H^∞ to the algebra of all operators. If the homomorphism has a nontrivial kernel, then T is said to belong to the class C_0, which turns out to have many of the properties of the set of operators on finite dimensional spaces. (For example, every operator in C_0 has a minimal function analogous to the minimal polynomial of a matrix.) A major success of the theory is obtained for the class of C_0 operators satisfying the additional requirement that $1-T^*T$ has finite rank. There is a structure theorem for these operators (also due to Sz.-Nagy and Foiaş) which parallels the Jordan canonical form in a very satisfying manner.

Evidently a large portion of this theory is the work of Sz.-Nagy and Foiaş, who have presented an extensive account of most of it in their volume, *Harmonic analysis of operators on Hilbert space* (North-Holland, Amsterdam; American Elsevier, New York; Akad. Kiado, Budapest, 1970). Douglas' article, proceeding at a rather lively pace, gives the reader an excellent overview of a considerable portion of this territory and makes a fine introduction to the larger work. A fairly large number of misprints in the article should keep the reader alert. (I counted fourteen, mostly trivial. Some of the more troubling ones occur in the proof of Proposition 3.9, where it should also be required that $|\hat{\phi}_i(\lambda)|=1$ on the complement of the set where it is required to be less than one.) For a different view of much of the same ground the reader may also wish to consult the expository article

Unitary dilations of Hilbert space operators and related topics by B. Sz.-Nagy, CBMS No. 19. Amer. Math. Soc., Providence, R. I., 1974.

In 1972 V. I. Lomonosov discovered a technique which settled the longstanding problem of whether or not two commutative compact operators have a common invariant subspace. He actually proved more: If A is a compact operator, then A shares a common invariant subspace with every operator that commutes with it. In fact as he asserted, a slight modification of his proof shows that the conclusion holds if A merely commutes with a compact operator. His technique, which utilizes the Schauder fixed point theorem, was immediately seized upon by many people and used to produce even stronger invariant subspace theorems. The paper *A survey of the Lomonosov technique in the theory of invariant subspaces*, by C. Pearcy and Allen L. Shields takes us through Lomonosov's contribution to some of its consequences and also discusses the current interesting state of the invariant subspace problem. A goodly portion of this material has also appeared in the monograph *Invariant subspaces* of H. Radjavi and P. Rosenthal, Springer-Verlag, Berlin, 1973.

There is much of interest in this book. The writing is generally brisk and meets a high standard for mathematical exposition. These essays can contribute a great deal to showing students some of the areas of operator theory that have been and are still the subject of considerable research.

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Model theory, by C. C. Chang and H. J. Keisler, Studies in Logic and the Foundations of Mathematics, Vol. 73, North-Holland, Amsterdam, 1973, xii+550 pp., $26.50.

1. **General remarks.**[1] This, in many ways remarkable, book is the first attempt at a systematic exposition of a young discipline, model theory, written by two of the main contributors to the subject. Naturally, the reviewer felt tempted to seize the opportunity to give a general discussion of the subject itself but unfortunately most of his general remarks had to be eliminated to bring the review down to a size acceptable to the Editors. To appreciate another difficulty of writing this review, consider one of the most striking features of the book, and in fact of model theory itself, namely the immense variety of topics, methods and orientation. One could hardly find two subjects further apart than, e.g. Artin's conjecture on p-adic number fields on the one hand, and the theory of measurable cardinals on the other, both given full expositions in the book. And these are just two examples of the large number of similarly disparate (at least, apparently disparate) matters in

[1] The reviewer would like to express his thanks to Stephen Garland, Victor Harnik, Jan Mycielski, Gonzalo Reyes, H. Jerome Keisler and Allan Swett for their helpful criticism of the original version of this review.